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Research Article

Mappings contracting perimeters of triangles in perturbed metric spaces

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ABSTRACT. In the present paper, we introduce the notion of mappings contracting perimeters of triangles in perturbed metric spaces, which we call perturbed mappings contracting perimeters of triangles. We provide a fixed point result for such mappings. We illustrate that our results are more general with some examples.

Keywords: Fixed point, mappings contracting triangles, perturbed metric spaces.

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1. INTRODUCTION AND PRELIMINARIES

Since Banach's pioneering fixed-point result was introduced (see [1]), there has been a sustained effort within the research community to derive more general results. There are two main directions of research: one is to relax the contractive conditions (see, for example, [3, 4, 6, 10, 13, 15, 16, 19, 22, 20] and the references therein), another area of interest has been altering the topological structure in which fixed-point results are established (see, for instance, [5, 7, 11, 12, 21], and the references therein).

In the present paper, we introduce the notion of mappings contracting perimeters of triangles in perturbed metric spaces, which we call perturbed mappings contracting perimeters of triangles. The new results combine the ideas of perturbed metric structures, recently proposed by Jleli and Samet in [8], with mappings contracting perimeters of triangles introduced by Petrov in [16].

We establish a fixed point theorem for such mappings in complete perturbed metric spaces, showing that the existence of fixed points can be guaranteed under a suitable contractive condition involving a three point condition. To highlight the applicability and generality of our approach, we provide an example demonstrating that our fixed point result remains valid in situations where classical contractive conditions fail. The framework proposed here opens new directions for further exploration of fixed point theory in generalized and perturbed metric spaces.

Recently, Jleli and Samet introduced a novel framework in [8], termed *perturbed metric spaces*, which extends the traditional concept of a metric space as follows:

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Definition 1.1. *Let $D, P : X \times X \to [0, \infty)$ be two given mappings. D is a perturbed metric on X with respect to P if the function*

$$d = D - P : X \times X \to \mathbb{R}, \quad (x, y) \mapsto D(x, y) - P(x, y)$$

is a metric on X. This means that for all $x, y, z \in X$, the following conditions hold:

(i) $(D - P)(x, y) \geq 0$,
(ii) $(D - P)(x, y) = 0$ *if and only if* $x = y$,
(iii) $(D - P)(x, y) = (D - P)(y, x)$,
(iv) $(D - P)(x, y) \leq (D - P)(x, z) + (D - P)(z, y)$.

We call P a perturbed mapping, $d = D - P$ the exact metric, and (X, D, P) a perturbed metric space.

Definition 1.2. *Let (X, D, P) be a perturbed metric space, $\{z_n\}$ a sequence in X, and $T : X \to X$.*

(i) We say that $\{z_n\}$ is a perturbed convergent sequence in (X, D, P) if $\{z_n\}$ is convergent in the metric space (X, d), where $d = D - P$ is the exact metric.
(ii) We say that $\{z_n\}$ is a perturbed Cauchy sequence in (X, D, P) if $\{z_n\}$ is a Cauchy sequence in the metric space (X, d).
(iii) We say that (X, D, P) is a complete perturbed metric space if (X, d) is a complete metric space, or equivalently, if every perturbed Cauchy sequence in (X, D, P) is a perturbed convergent sequence in (X, D, P).
(iv) We say that T is a perturbed continuous mapping if T is continuous with respect to the exact metric d.

In [8], a generalization of Banach's fixed point theorem is proved:

Theorem 1.1. *Let (X, D, P) be a complete perturbed metric space and $T : X \to X$ a given mapping. Assume that the following conditions hold:*

(i) T is a perturbed continuous mapping.
(ii) There exists $\lambda \in (0, 1)$ such that

$$D(Tx, Ty) \leq \lambda D(x, y) \quad \text{for all } x, y \in X.$$

Then, T admits a unique fixed point.

Very recently, Petrov introduced in [16] a new type of mappings called mappings contracting perimeters of triangles, which are a three-point analogue of Banach contractions [1]:

Definition 1.3 (Petrov [16])**.** *Let (X, d) be a metric space with $|X| \geq 3$. We shall say that $T \colon X \to X$ is a mapping contracting perimeters of triangles on X if there exists $\alpha \in [0, 1)$ such that the inequality*

$$d(Tx, Ty) + d(Ty, Tz) + d(Tz, Tx) \leq \alpha[d(x, y) + d(y, z) + d(z, x)],$$

holds for all three pairwise distinct points $x, y, z \in X$.

Petrov proved in [16] a fixed point theorem for this kind of mapping:

Theorem 1.2 (Petrov [16])**.** *Let (X, d), $|X| \geq 3$ be a complete metric space and let $T \colon X \to X$ be a mapping contracting perimeters of triangles on X. Then, T has a fixed point if and only if T does not possess periodic points of prime period 2. The number of fixed points is at most 2.*

The newly introduced mappings were further studied and extended in [2, 9, 14, 17, 18, 19, 15, 24].

2. Main results

Definition 2.4. *Let (X, D, P) be a perturbed metric space with $|X| \geq 3$. We shall say that $T\colon X \to X$ is a perturbed mapping contracting perimeters of triangles if there exists $\lambda \in [0,1)$ such that the inequality*

$$D(Tx,Ty) + D(Ty,Tz) + D(Tz,Tx) \leq \lambda[D(x,y) + D(y,z) + D(z,x)], \tag{2.1}$$

holds for all three pairwise distinct points $x, y, z \in X$.

Theorem 2.3. *Let (X, D, P) be a complete perturbed metric space and $T : X \to X$ such that*

(i) *$T(Tx) \neq x$ for all $x \in X$ such that $Tx \neq x$.*
(ii) *T is a perturbed continuous mapping contracting perimeters of triangles.*

Then, T has a fixed point. The number of fixed points is at most two.

Proof. Let $x_0 \in X$, arbitrarily chosen, but fixed and the Picard iteration

$$x_{n+1} = Tx_n, \quad \forall n \geq 0.$$

We shall show that T has at least one fixed point. Suppose that x_n is not a fixed point of the mapping T for every $n = 0, 1, \dots$ Then, we have $x_n = Tx_{n-1} \neq x_{n-1}$ and $x_{n+1} = T(Tx_{n-1}) \neq x_{n-1}$ for every $n = 1, 2, \dots$. Hence, by condition (i), x_{n-1}, x_n and x_{n+1} are pairwise distinct.

Let

$$d_n = D(x_n, x_{n+1}) + D(x_{n+1}, x_{n+2}) + D(x_{n+2}, x_n).$$

Since x_{n-1}, x_n and x_{n+2} are pairwise distinct, by (2.1) we have

$$D(x_{n+1},x_{n+2})+D(x_{n+2},x_{n+3})+D(x_{n+3},x_{n+1}) \leq \alpha[D(x_n,x_{n+1})+D(x_{n+1},x_{n+2})+D(x_{n+2},x_n)],$$

i.e.

$$d_{n+1} \leq \alpha d_n,$$

and inductively we obtain

$$d_{n+1} \leq \alpha^{n+1} d_0.$$

It is clear that $D(x_{n+1}, x_{n+2}) \leq d_n$, so we obtain

$$D(x_{n+1}, x_{n+2}) \leq d_n \leq \alpha^n d_0.$$

Now, let $d = D - P$ be the exact metric. Then, from the above inequity, we obtain that

$$d(x_n, x_{n+1}) + P(x_n, x_{n+1}) \leq \alpha^n d_0, \quad \forall n \geq 0.$$

Since $d(x_n, x_{n+1}) \leq d(x_n, x_{n+1}) + P(x_n, x_{n+1})$, we get that

$$d(x_n, x_{n+1}) \leq \alpha^n d_0, \quad \forall n \geq 0.$$

Then,

$$\begin{aligned} d(x_n, x_{n+p}) &\leq \alpha^n d_0 + \alpha^{n+1} d_0 + \cdots + \alpha^{n+p-1} d_0 \\ &\leq \frac{\alpha^n}{1-\alpha} d_0. \end{aligned}$$

Since $\alpha \in [0,1)$ we obtain that $\{x_n\}$ is a Cauchy sequence in the metric space (X, d), so $\{x_n\}$ is a perturbed Cauchy sequence in (X, D, P). Thus, by completeness of the perturbed metric space (X, D, P), there exists $x^* \in X$ such that

$$\lim_{n\to\infty} d(x_n, x^*) = 0.$$

To show that $Tx^* = x^*$, since T is a perturbed continuous mapping, it follows that

$$\lim_{n\to\infty} d(Tx_n, Tx^*) = 0,$$

which implies

$$\lim_{n\to\infty} d(x_{n+1}, Tx^*) = 0.$$

Since $d = D - P$ is a metric on X, by uniqueness of the limit we obtain $x^* = Tx^*$, i.e. x^* is a fixed point of T.

Finally, if there exist at least three pairwise distinct fixed points x^*, y^* and z^*, then $Tx^* = x^*$, $Ty^* = y^*$ and $Tz^* = z^*$, which contradicts (2.1). Therefore, the number of fixed points is at most two.

□

Let us provide an example of perturbed mappings that contracting triangles.

Example 2.1. *Let $X = \{0, 1, 2\}$, and define the mappings: Let $X = \{0, 1, 2\}$, and define:*

$$d(x,y) = \begin{cases} 1, & \text{if } x \neq y, \\ 0, & \text{if } x = y, \end{cases} \qquad P(x,y) = \begin{cases} 0.1, & \text{if } \{x,y\} = \{1,2\}, \\ 0, & \text{otherwise}. \end{cases}$$

Let $D(x, y) = d(x, y) + P(x, y)$.

(X, D, P) is a complete perturbed metric space since (X, d) is complete.

Define the mapping $T : X \to X$ by

$$T(0) = 0, \quad T(1) = 0, \quad T(2) = 2.$$

We have

$$\begin{aligned} D(T0, T1) + D(T1, T2) + D(T2, T0) &= D(0, 0) + D(0, 2) + D(2, 0) \\ &= 0 + 1.1 + 1.1 = 2.2, \\ D(0, 1) + D(1, 2) + D(2, 0) &= 1 + 1.1 + 1 = 3.1. \end{aligned}$$

Hence,

$$D(Tx, Ty) + D(Ty, Tz) + D(Tz, Tx) \leq \lambda[D(x, y) + D(y, z) + D(z, x)]$$

with $\lambda = \frac{2.2}{3.1} \approx 0.71 < 1$. Thus, T is a perturbed mapping contracting triangles.

However, T is not a Banach since

$$D(T0, T2) = D(0, 2) = 1.1 = D(0, 2).$$

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