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Research Article

On new φ-fixed point results involving discontinuous control functions with the effectively example and its applications

PATHAITHEP KUMROD AND WUTIPHOL SINTUNAVARAT*

ABSTRACT. The main purpose of this paper is to extend and enhance the results of Karapinar *et al.* [7] by relaxing the continuity assumption on control functions in the contractive setting. The validity and wider applicability of our principal theorem are illustrated through examples. In addition, our generalized framework yields a homotopy result and establishes the existence of solutions for a class of integral equations.

Keywords: φ-fixed points, φ-Picard mappings, homotopy result.

2020 Mathematics Subject Classification: 47H10, 54H25.

1. INTRODUCTION AND PRELIMINARIES

In the last ten decades, the classical Banach Contraction Principle (shortened, BCP) in [4] has been investigated and improved by many researchers in several different ways, by the following ideas:

- introducing the generalized Banach contractive conditions;
- increasing the number of involved mappings;
- extending the class of ambient spaces;
- enlarging the idea of fixed points.

Nowadays, there are many interesting research works in all of the above directions (see [11, 1, 8, 2, 10] and the references therein). In the following discussion, we focus particularly on those studies that have directly inspired the development of the present work and are closely related to its main ideas.

In 1969, Boyd and Wong [5] proved the fixed point theorem, which is one of the interesting generalizations of the classical Banach contraction principle, and introduced the following family of control functions:

$$\Psi = \left\{ \psi : [0, \infty) \to [0, \infty) \mid \psi(t) < t \text{ for each } t > 0 \text{ and } \limsup_{r \to t^+} \psi(r) < t \text{ for each } t > 0 \right\}.$$

Afterwards, many mathematicians proved various fixed point results with the help of the control functions in Ψ.

In another direction, Jleli et al. [6] extended the Banach contraction principle by introducing new control functions and first proposed the notions of φ-fixed points and φ-Picard mappings. Before presenting their definitions and main results, we recall the following essential notions.

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*Corresponding author: Wutiphol Sintunavarat; wutiphol@mathstat.sci.tu.ac.th
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Let X be a nonempty set, $\varphi : X \to [0, \infty)$ be a given function, and $T : X \to X$ be a mapping. We denote by F_T the set of all fixed points of T, and by Z_φ the set of all zeros of the function φ, that is,

$$Z_\varphi = \{x \in X \mid \varphi(x) = 0\}.$$

Definition 1.1 ([6]). *Let X be a nonempty set and $\varphi : X \to [0, \infty)$ be a given function. An element $z \in X$ is called a φ-fixed point of $T : X \to X$ if and only if z is a fixed point of T and $\varphi(z) = 0$, that is, $z \in F_T \cap Z_\varphi$.*

Definition 1.2 ([6]). *Let (X, d) be a metric space and $\varphi : X \to [0, \infty)$ be a given function. A mapping $T : X \to X$ is called a φ-Picard mapping if and only if the following conditions hold:*

(i) *$F_T \cap Z_\varphi = \{z\}$, where $z \in X$;*
(ii) *$T^n x \to z$ as $n \to \infty$ for each $x \in X$.*

In the sequel, we denote by $\mathcal{F}$ the class of all functions $F : [0, \infty)^3 \to [0, \infty)$ satisfying the following conditions:

(F1) $\max\{a, b\} \leq F(a, b, c)$ for all $a, b, c \in [0, \infty)$;
(F2) $F(0, 0, 0) = 0$;
(F3) F is continuous.

This class of functions was first introduced by Jleli et al. [6] to generalize classical contractive conditions and to unify several fixed point results under a broader framework. To illustrate the definition, we present some typical examples below, followed by Jleli et al. [6].

Example 1.1 ([6]). *The following functions $F_1, F_2, F_3 : [0, \infty)^3 \to [0, \infty)$ belong to $\mathcal{F}$:*

(i) *$F_1(a, b, c) = a + b + c$ for all $a, b, c \in [0, \infty)$;*
(ii) *$F_2(a, b, c) = \max\{a, b\} + c$ for all $a, b, c \in [0, \infty)$;*
(iii) *$F_3(a, b, c) = a + a^2 + b + c$ for all $a, b, c \in [0, \infty)$.*

These examples demonstrate that the class $\mathcal{F}$ encompasses a wide range of control functions, each leading to different types of contractive mappings. Using this framework, Jleli et al. [6] established the following fundamental fixed point theorem.

Theorem 1.1 ([6]). *Let (X, d) be a complete metric space, and let $\varphi : X \to [0, \infty)$ be a given lower semi-continuous function. Suppose that $T : X \to X$ is a mapping satisfying*

$$F(d(Tx, Ty), \varphi(Tx), \varphi(Ty)) \leq k\, F(d(x, y), \varphi(x), \varphi(y)) \tag{1.1}$$

for all $x, y \in X$, where $F \in \mathcal{F}$ and $k \in [0, 1)$. Then T is a φ-Picard mapping and $F_T \subseteq Z_\varphi$.

If we take $F(a, b, c) = a+b+c$ for all $a, b, c \in [0, \infty)$ and $\varphi(x) = 0$ for all $x \in X$ in Theorem 1.1, then (1.1) reduces to the classical Banach contractive condition. Consequently, Theorem 1.1 also reduces to the well-known Banach contraction principle.

Motivated by both the works of Boyd and Wong [5] and Jleli et al. [6], Karapinar et al. [7] proved φ-fixed point theorems by replacing the condition (F2) with the following modified assumption:

(F2*) $F(a, 0, 0) = a$ for all $a \geq 0$,

and by substituting the constant k with a control function $\psi \in \Psi$. This generalization significantly broadens the applicability of the φ-Picard mapping framework. In 2017, Asadi [3] further refined these results by showing that the continuity of $F \in \mathcal{F}$ in Theorem 1.1 can be weakened. In particular, the theorem remains valid when the following weaker requirement replaces the continuity condition:

- $\limsup_{n\to\infty} F(x_n, y_n, 0) \leq F(x, y, 0)$ whenever $x_n \to x$ and $y_n \to y$ as $n \to \infty$.

Example 1.2 ([3]). *Let $F : [0,\infty)^3 \to [0,\infty)$ be defined by $F(a,b,c) = a + b + [c]$ or $F(a,b,c) = \max\{a,b\} + [c]$ for all $a,b,c \in [0,\infty)$, where $[c]$ denotes the integer part of c. Then F satisfies the condition of Asadi* [3]*, but F is not continuous.*

Motivated by the results of Asadi [3] and Karapinar et al. [7], the main objective of this work is to introduce a new class of control functions used in a generalized contractive condition and to establish a φ-fixed point theorem for mappings satisfying this condition with respect to the proposed class of control functions. Our results properly extend and generalize various well-known fixed point theorems in the existing literature. Additionally, illustrative examples and applications are provided to demonstrate the validity and practical utility of the obtained results.

2. Main results

First, we denote by $\mathcal{G}$ the set of all functions $G : [0,\infty)^3 \to [0,\infty)$ that satisfy the following conditions:

(G1) $\max\{a,b\} \le G(a,b,c)$ for all $a,b,c \in [0,\infty)$,
(G2) $G(a,0,0) = a$ for all $a \ge 0$,
(G3) $\limsup_{n\to\infty} G(x_n,y_n,z_n) \le G(x,0,0)$ whenever $x_n \to x$, $y_n \to 0$, and $z_n \to 0$ as $n \to \infty$.

As examples, consider the functions $G_1, G_2 : [0,\infty)^3 \to [0,\infty)$ defined by

$$G_1(a,b,c) = a + b + [c] \quad \text{and} \quad G_2(a,b,c) = \max\{a,b\} + [c]$$

for all $a,b,c \in [0,\infty)$, where $[c]$ denotes the integer part of c. Clearly, $G_1, G_2 \in \mathcal{G}$.

Next, we introduce a new class of control functions that will be used to define a generalized contractive condition in our main φ-fixed point theorem, extending the class ψ considered in previous works.

Let Λ denote the set of all functions $\lambda : [0,\infty) \to [0,\infty)$ satisfying:

(λ1) $\lambda(t) < t$ for all $t > 0$;
(λ2) if $\{a_n\}$ is a sequence in $[0,\infty)$ with $\limsup_{n\to\infty} a_n \le a$, then $\limsup_{n\to\infty} \lambda(a_n) \le \lambda(a)$.

We are now ready to present our main result.

Theorem 2.2. *Let (X,d) be a complete metric space and T be a self mapping on X such that*

$$G(d(Tx,Ty),\varphi(Tx),\varphi(Ty)) \le \lambda(G(d(x,y),\varphi(x),\varphi(y))) \tag{2.2}$$

for all $x,y \in X$, where $\varphi : X \to [0,\infty)$ is lower semi-continuous, $G \in \mathcal{G}$, and $\lambda \in \Lambda$. Then $F_T \subseteq Z_\varphi$ and T is a φ-Picard mapping.

Proof. First, we need to show that $F_T \subseteq Z_\varphi$. Let $x \in F_T$. By putting $x = y$ in (2.2), we have

$$G(0,\varphi(x),\varphi(x)) \le \lambda(G(0,\varphi(x),\varphi(x))). \tag{2.3}$$

Assume on the contrary that $\varphi(x) > 0$. Then $G(0,\varphi(x),\varphi(x)) > 0$. From (2.3) and ($\lambda$1), we get

$$G(0,\varphi(x),\varphi(x)) \le \lambda(G(0,\varphi(x),\varphi(x))) < G(0,\varphi(x),\varphi(x)),$$

which is a contradiction. Hence,

$$\varphi(x) = 0,$$

which implies that

$$F_T \subseteq Z_\varphi. \tag{2.4}$$

Next, we will show that T is a φ-Picard mapping. Let x_0 be an arbitrary point in X. Define the sequence $\{x_n\} \subseteq X$ by $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$. If $x_{n^*} = x_{n^*-1}$ for some $n^* \in \mathbb{N}$, then x_{n^*-1} is a fixed point of T. We have nothing to prove and so we may assume that

$$d(x_n, x_{n-1}) > 0 \tag{2.5}$$

for all $n \in \mathbb{N}$. It follows from (G1) that

$$G(d(x_n, x_{n-1}), \varphi(x_n), \varphi(x_{n-1})) > 0 \tag{2.6}$$

for all $n \in \mathbb{N}$. Hence by the contractive condition (2.2), (2.6) and $(\lambda 1)$ we have

$$\begin{aligned} G(d(x_{n+1}, x_n), \varphi(x_{n+1}), \varphi(x_n)) &\leq \lambda(G(d(x_n, x_{n-1}), \varphi(x_n), \varphi(x_{n-1}))) \\ &< G(d(x_n, x_{n-1}), \varphi(x_n), \varphi(x_{n-1})) \end{aligned} \tag{2.7}$$

for all $n \in \mathbb{N}$. This shows that $\{G(d(x_{n+1}, x_n), \varphi(x_{n+1}), \varphi(x_n))\}$ is a decreasing sequence and hence it converges to some point $r \geq 0$, that is,

$$\lim_{n\to\infty} G(d(x_{n+1}, x_n), \varphi(x_{n+1}), \varphi(x_n)) = r. \tag{2.8}$$

From (2.7), (2.8) and the squeeze theorem, we get

$$\lim_{n\to\infty} \lambda(G(d(x_n, x_{n-1}), \varphi(x_n), \varphi(x_{n-1}))) = r. \tag{2.9}$$

We will show that $r = 0$. Suppose by way of contradiction that $r > 0$. By $(\lambda 1)$, $(\lambda 2)$, (2.8) and (2.9) we have

$$r = \limsup_{n\to\infty} \lambda(G(d(x_n, x_{n-1}), \varphi(x_n), \varphi(x_{n-1}))) \leq \lambda(r) < r,$$

which provides a contradiction. Therefore, $r = 0$, that is,

$$\lim_{n\to\infty} G(d(x_{n+1}, x_n), \varphi(x_{n+1}), \varphi(x_n)) = \lim_{n\to\infty} \lambda(G(d(x_n, x_{n-1}), \varphi(x_n), \varphi(x_{n-1}))) = 0,$$

and thus, by (G1)

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = \lim_{n\to\infty} \varphi(x_{n+1}) = 0. \tag{2.10}$$

In what follows, we shall prove that $\{x_n\}$ is a Cauchy sequence. Suppose, to the contrary, that $\{x_n\}$ is not a Cauchy sequence. Then there exists $\epsilon > 0$ such that we can find subsequences $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ of $\{x_n\}$ with $n(k) > m(k) \geq k$, for all positive integer k, satisfying

$$d\left(x_{m(k)}, x_{n(k)}\right) \geq \epsilon. \tag{2.11}$$

We may assume that

$$d\left(x_{m(k)}, x_{n(k)-1}\right) < \epsilon, \tag{2.12}$$

by choosing $n(k)$ to be the smallest integer exceeding $m(k)$ for which (2.11) holds. Now, using the triangle inequality and (2.12), we have

$$\epsilon \leq d\left(x_{m(k)}, x_{n(k)}\right) \leq d\left(x_{m(k)}, x_{n(k)-1}\right) + d\left(x_{n(k)-1}, x_{n(k)}\right) < \epsilon + d\left(x_{n(k)-1}, x_{n(k)}\right).$$

Letting $k \to \infty$ in the previous inequality and using (2.10), we get

$$\lim_{k\to\infty} d\left(x_{m(k)}, x_{n(k)}\right) = \epsilon. \tag{2.13}$$

Using $(G2)$, $(G3)$, (2.10) and (2.13), it follows that

$$\limsup_{k\to\infty} G\left(d\left(x_{m(k)}, x_{n(k)}\right), \varphi\left(x_{m(k)}\right), \varphi\left(x_{n(k)}\right)\right) \leq G(\epsilon, 0, 0) = \epsilon. \tag{2.14}$$

From $(\lambda 2)$, it follows from (2.14) that

$$\limsup_{k\to\infty} \lambda\left(G\left(d\left(x_{m(k)}, x_{n(k)}\right), \varphi\left(x_{m(k)}\right), \varphi\left(x_{n(k)}\right)\right)\right) \leq \lambda(\epsilon). \tag{2.15}$$

On the other hand, by the triangle inequality, (2.2) and $(G1)$, for all $k \in \mathbb{N}$, we see that

$$\begin{aligned}\epsilon &\leq d\left(x_{m(k)}, x_{n(k)}\right) \\ &\leq d\left(x_{m(k)}, x_{m(k)+1}\right) + d\left(x_{m(k)+1}, x_{n(k)+1}\right) + d\left(x_{n(k)+1}, x_{n(k)}\right) \\ &\leq d\left(x_{m(k)}, x_{m(k)+1}\right) + G\left(d\left(x_{m(k)+1}, x_{n(k)+1}\right), \varphi\left(x_{m(k)+1}\right), \varphi\left(x_{n(k)+1}\right)\right) + d\left(x_{n(k)+1}, x_{n(k)}\right) \\ &\leq d\left(x_{m(k)}, x_{m(k)+1}\right) + \lambda\left(G\left(d(x_{m(k)}, x_{n(k)}), \varphi(x_{m(k)}), \varphi(x_{n(k)})\right)\right) + d\left(x_{n(k)+1}, x_{n(k)}\right).\end{aligned}$$

Taking limit superior as $k \to \infty$ and using (2.10), (2.15) and $(\lambda 2)$, we have

$$\epsilon \leq \lambda(\epsilon) < \epsilon,$$

which is a contradiction. Therefore, $\{x_n\}$ is a Cauchy sequence. By the completeness of X, there exists a point $z \in X$ such that

$$\lim_{n\to\infty} x_n = z. \tag{2.16}$$

Using (2.10), (2.16) and the semi-continuity of φ, we get

$$0 \leq \varphi(z) \leq \liminf_{n\to\infty} \varphi(x_n) = 0,$$

which implies that

$$\varphi(z) = 0. \tag{2.17}$$

Now, we shall prove that z is a fixed point of T. From (2.5) and (2.16), there exists a subsequence $\{f(n)\}$ of $\{n\}$ such that

$$d\left(x_{f(n)}, z\right) > 0 \tag{2.18}$$

holds for any $n \in \mathbb{N}$. By applying (2.2) and using $(G1)$, $(\lambda 1)$, (2.17) and (2.18), we have

$$\begin{aligned} d\left(x_{f(n)+1}, Tz\right) &\leq \max\left\{d\left(x_{f(n)+1}, Tz\right), \varphi\left(x_{f(n)+1}\right)\right\} \\ &\leq G\left(d\left(x_{f(n)+1}, Tz\right), \varphi\left(x_{f(n)+1}\right), \varphi(Tz)\right) \\ &\leq \lambda\left(G\left(d\left(x_{f(n)}, z\right), \varphi\left(x_{f(n)}\right), \varphi(z)\right)\right) \\ &< G\left(d\left(x_{f(n)}, z\right), \varphi\left(x_{f(n)}\right), \varphi(z)\right) \\ &= G\left(d\left(x_{f(n)}, z\right), \varphi\left(x_{f(n)}\right), 0\right). \end{aligned}$$

Thus

$$\limsup_{n\to\infty} d\left(x_{f(n)+1}, Tz\right) \leq \limsup_{n\to\infty} G\left(d\left(x_{f(n)}, z\right), \varphi\left(x_{f(n)}\right), 0\right) \leq G(0,0,0) = 0,$$

which implies that

$$\lim_{n\to\infty} d(x_{f(n)+1}, Tz) = 0.$$

By the uniqueness of the limit, we get $z = Tz$.

Finally, we prove that z is the unique fixed point of T. Assume, for the sake of contradiction, that there exists another fixed point $w \in X$ such that $z \neq w$. Then $z = Tz$ and $w = Tw$. Using (2.2), (2.4), $(\lambda 1)$, and $(G2)$, we have

$$d(z,w) = G(d(z,w),0,0) \leq \lambda(G(d(z,w),0,0)) < G(d(z,w),0,0) = d(z,w),$$

which is a contradiction. Hence, z is the unique fixed point of T. From this conclusion, we can conclude that T is a φ-Picard mapping. □

Remark 2.1. *Note that in the proof of Theorem 2.2, we use the fact that $a > 0$ or $b > 0$ implies that $G(a,b,c) > 0$ for all $a,b,c > 0$. So our proof is simpler and shorter than the proof of Karapinar et al. [7].*

Remark 2.2. *By the properties of the classes $\mathcal{G}$ and Λ, we immediately obtain the main theorem of Karapinar et al.* [7].

The following example shows that our theorem is a proper generalization of some results in the literature.

Example 2.3. *Let $X = [0, 3]$ endowed with the usual metric $d : X \times X \to [0, \infty)$ defined by $d(x, y) = |x - y|$ for all $x, y \in X$. Then (X, d) is a complete metric space. Define the self-mapping $T : X \to X$ by*

$$Tx = \begin{cases} \frac{x^2}{9}, & 0 \leq x < 2, \\ \frac{x}{x+2}, & 2 \leq x \leq 3. \end{cases}$$

Furthermore, we define a function $\lambda : [0, \infty) \to [0, \infty)$ by

$$\lambda(t) = \begin{cases} \frac{t}{3}, & 0 \leq t < 2, \\ \sin\left(\frac{2}{t}\right) + \frac{9}{8}, & t \geq 2. \end{cases}$$

The graph of λ shown in blue is given in Figure 1. It is easy to see that $\lambda \in \Lambda \cap \Psi$.

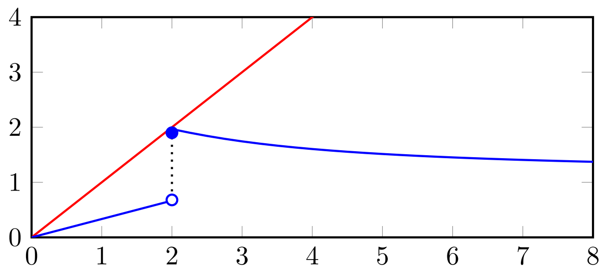

FIGURE 1. The graph of λ

Note that the Banach contraction principle is not applicable because T is not continuous at a point 2. Also, the fixed point theorem of Boyd and Wong [5] *cannot be applied in this case. Indeed, for $x = 1$ and $y = 2$, we get*

$$d(Tx, Ty) = |T1 - T2| = \left|\frac{1}{9} - \frac{1}{2}\right| = \frac{7}{18} \not< \frac{1}{3} = \lambda(1) = \lambda(d(1, 2)) = \lambda(d(x, y)).$$

Now, let us consider the mapping $G : [0, \infty)^3 \to [0, \infty)$ and $\varphi : X \to [0, \infty)$ defined by

$$G(a, b, c) = a + b + [c]$$

for each $a, b, c \geq 0$, where $[c]$ is the integer part of c and $\varphi(x) = \frac{x}{2}$ for each $x \in X$. Clearly, $G \in \mathcal{G}$ and φ is lower semi-continuous. Finally, we shall claim that the mapping T satisfies the contractive condition (2.2). Suppose that $x, y \in X$. We distinguish the following four cases:

Case 1: *If* $(x, y) \in [0, 2) \times [0, 2)$*, then*

$$\begin{aligned} & G(d(Tx,Ty),\varphi(Tx),\varphi(Ty)) \\ = \;& d(Tx,Ty)+\varphi(Tx)+[\varphi(Ty)] \\ = \;& \frac{|x^2-y^2|}{9}+\frac{x^2}{18} \\ \le \;& \frac{|x-y|}{3}+\frac{x}{6} \\ \le \;& \lambda\left(|x-y|+\frac{x}{2}+\left[\frac{y}{2}\right]\right) \\ = \;& \lambda(d(x,y)+\varphi(x)+[\varphi(y)]) \\ = \;& \lambda(G(d(x,y),\varphi(x),\varphi(y))). \end{aligned} \tag{2.19}$$

The 3D graphs (plotted in MATLAB) in Figure 2 guarantee that the condition (2.19) holds.

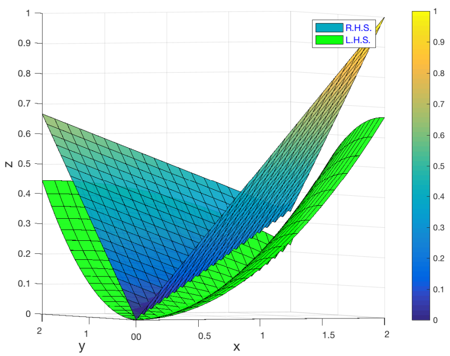

FIGURE 2. The graph of (2.19)

Case 2: *If* $(x, y) \in [2, 3] \times [2, 3]$*, then*

$$\begin{aligned} & G(d(Tx,Ty),\varphi(Tx),\varphi(Ty)) \\ = \;& d(Tx,Ty)+\varphi(Tx)+[\varphi(Ty)] \\ = \;& \left|\frac{x}{x+2}-\frac{y}{y+2}\right|+\frac{x}{2(x+2)} \\ \le \;& \sin\left(\frac{2}{|x-y|+\frac{x}{2}+\left[\frac{y}{2}\right]}\right)+\frac{9}{8} \\ = \;& \lambda\left(|x-y|+\frac{x}{2}+\left[\frac{y}{2}\right]\right) \\ = \;& \lambda(d(x,y)+\varphi(x)+[\varphi(y)]) \\ = \;& \lambda(G(d(x,y),\varphi(x),\varphi(y))). \end{aligned} \tag{2.20}$$

The 3D graphs (plotted in MATLAB) in Figure 3 guarantee that the condition (2.20) holds.

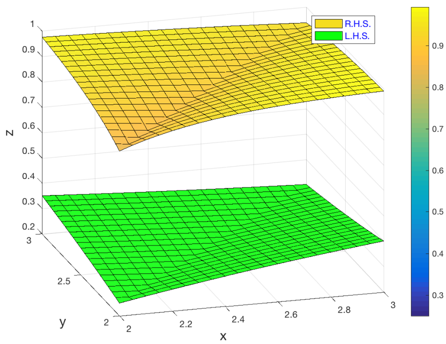

FIGURE 3. The graph of (2.20)

Case 3: *If $x \in [0,2)$ and $y \in [2,3]$, then*

$$
\begin{aligned}
& G(d(Tx,Ty),\varphi(Tx),\varphi(Ty)) \\
= \; & d(Tx,Ty)+\varphi(Tx)+[\varphi(Ty)] \\
= \; & \left|\frac{x^2}{9}-\frac{y}{y+2}\right|+\frac{x^2}{18} \\
\le \; & \frac{|x-y|}{3}+\frac{x}{6}+\frac{1}{3}\left[\frac{y}{2}\right] \\
= \; & \lambda\left(|x-y|+\frac{x}{2}+\left[\frac{y}{2}\right]\right) \\
= \; & \lambda(d(x,y)+\varphi(x)+[\varphi(y)]) \\
= \; & \lambda(G(d(x,y),\varphi(x),\varphi(y))).
\end{aligned} \tag{2.21}
$$

The 3D graphs (plotted in MATLAB) in Figure 4 guarantee that the condition (2.21) holds.

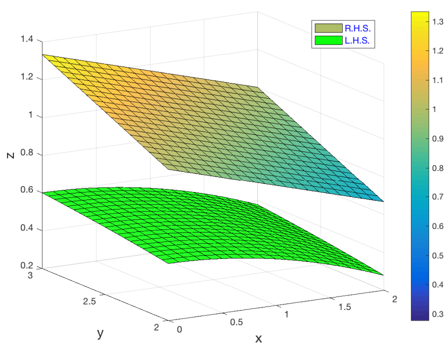

FIGURE 4. The graph of (2.21)

Case 4: *If $x \in [2,3]$ and $y \in [0,2)$, then*

$$\begin{aligned} & G(d(Tx,Ty),\varphi(Tx),\varphi(Ty)) \\ = & \ d(Tx,Ty)+\varphi(Tx)+[\varphi(Ty)] \\ = & \left|\frac{x}{x+2}-\frac{y^2}{9}\right|+\frac{x}{2(x+2)} \\ \le & \ \frac{|x-y|}{3}+\frac{x}{6} \\ = & \ \lambda\left(|x-y|+\frac{x}{2}\right) \\ = & \ \lambda(d(x,y)+\varphi(x)+[\varphi(y)]) \\ = & \ \lambda(G(d(x,y),\varphi(x),\varphi(y))). \end{aligned} \tag{2.22}$$

The 3D graphs (plotted in MATLAB) in Figure 5 guarantee that the condition (2.22) holds.

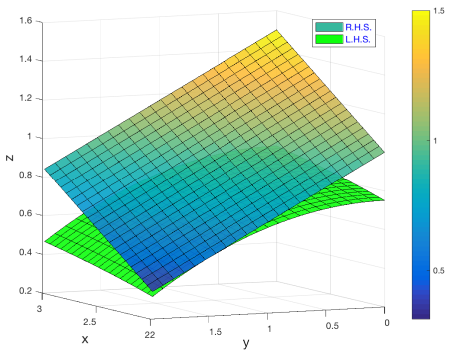

FIGURE 5. The graph of (2.22)

Considering all the above cases, we conclude that T satisfies the contractive condition (2.2). Therefore, by Theorem 2.2, T admits a unique φ-fixed point.

Corollary 2.1. *Let (X,d) be a complete metric space and $T : X \to X$ be a given mapping such that*

$$d(Tx,Ty) \le \lambda(d(x,y))$$

for all $x,y \in X$, where $\lambda \in \Lambda$. Thus, T has a unique fixed point.

Proof. The proof of this corollary immediate by taking $\varphi \equiv 0$ in Theorem 2.2 and using condition $(G2)$. □

Remark 2.3. *Based on the fact that Λ is wider than Ψ, Corollary 2.1 is the real proper extension of the fixed point theorem of Boyd and Wong [5]. However, if we take $\varphi \equiv 0$ in Theorem 2.1 of Karapinar et al. [7], we have that the obtained result is equivalent to the Boyd and Wong fixed point theorem. This follows the advantage of our main result in this work, which is supported by several results in the literature, as shown in Figure 6.*

3. APPLICATIONS

In this section, we propose two applications which is derived from the main φ-fixed point result in the previous section. These applications involve the analysis of the homotopy result and the analysis of solutions to nonlinear Volterra integral equations of the second kind.

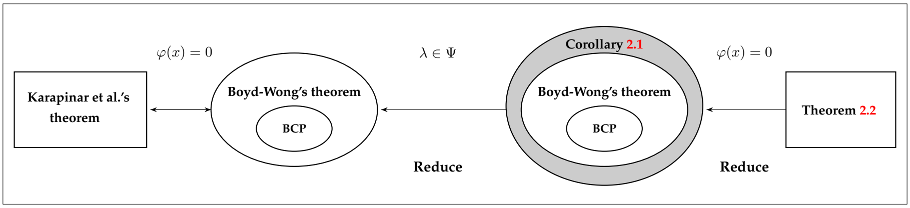

FIGURE 6. The difference of consequence between our theorem and Karapinar et al.'s theorem

3.1. **Application to the homotopy result.** In this part, we present the homotopic results that can be obtained from the φ-fixed point result in the previous section. We begin by introducing the new class of control functions used in the main result in this part. Denote by $\mathcal{G}^*$ the class of all functions $G \in \mathcal{G}$ satisfying the following property:

$(G4)$ for all $a, b, c, d \geq 0$,

$$a \leq c + d \implies G(a, b, 0) \leq G(c, b, 0) + d.$$

Example 3.4 ([7]). *Let $G_1, G_2, G_3 : [0, \infty)^3 \to [0, \infty)$ be defined by*

(i) $G_1(a, b, c) = (a + b)e^c$ *for all* $a, b, c \geq 0$;
(ii) $G_2(a, b, c) = (a + b)(c + 1)^n$ *for all* $a, b, c \geq 0$, *where* $n \in \mathbb{N}$;
(iii) $G_3(a, b, c) = ae^{c+b} + be^{a+c}$ *for all* $a, b, c \geq 0$.

Then, $G_1, G_2 \in \mathcal{G}^*$ *and* $\mathcal{G} \ni G_3 \notin \mathcal{G}^*$.

The following homotopy result can be derived from Theorem 2.2 and the same technique in the proof of Theorem 3.1 in [7].

Theorem 3.3. *Let (X, d) be a complete metric space, U be an open subset of X, and V be a closed subset of X with $U \subset V$. Suppose that $H : V \times [0, 1] \to X$ has the following properties:*

(C1) $x \neq H(x, \lambda)$ *for every* $x \in V \setminus U$ *and* $\lambda \in [0, 1]$;
(C2) *there exist a continuous function* $\varphi : X \to [0, \infty)$, $L \in (0, 1)$, *and* $G \in \mathcal{G}^*$ *such that*

$$G(d(H(x, \lambda), H(y, \lambda)), \varphi(H(x, \lambda)), \varphi(H(y, \lambda))) \leq LG(d(x, y), \varphi(x), \varphi(y))$$

for all $x, y \in V$ *and* $\lambda \in [0, 1]$;
(C3) *there exists a continuous function* $\eta : [0, 1] \to \mathbb{R}$ *such that*

$$G(d(H(x, \lambda), H(x, \mu)), \varphi(H(x, \lambda)), \varphi(H(x, \mu))) \leq |\eta(\lambda) - \eta(\mu)|$$

for all $x \in V$ *and* $\lambda, \mu \in [0, 1]$.

Then $H(\cdot, 0)$ *has a fixed point if and only if* $H(\cdot, 1)$ *has a fixed point.*

Based on the fact that the class $\mathcal{G}^*$ enlarges the class

$$\overline{\mathcal{G}} := \{G : [0, \infty)^3 \to [0, \infty) \mid G \text{ satisfies (G1), (G3), (G4) and } G \text{ is continuous}\}$$

which is defined in [7], we get the following result:

Corollary 3.2 (Theorem 3.1 in [7]). *Let (X, d) be a complete metric space, U be an open subset of X, and V be a closed subset of X with $U \subset V$. Suppose that $H : V \times [0, 1] \to X$ has the following properties:*

(C1) $x \neq H(x, \lambda)$ *for every* $x \in V \setminus U$ *and* $\lambda \in [0, 1]$;

(C2) *there exist a continuous function* $\varphi : X \to [0, \infty)$, $L \in (0, 1)$, *and* $G \in \overline{\mathcal{G}}$ *such that*

$$G(d(H(x, \lambda), H(y, \lambda)), \varphi(H(x, \lambda)), \varphi(H(y, \lambda))) \le LG(d(x, y), \varphi(x), \varphi(y))$$

for all $x, y \in V$ *and* $\lambda \in [0, 1]$;

(C3) *there exists a continuous function* $\eta : [0, 1] \to \mathbb{R}$ *such that*

$$F(d(H(x, \lambda), H(x, \mu)), \varphi(H(x, \lambda)), \varphi(H(x, \mu))) \le |\eta(\lambda) - \eta(\mu)|$$

for all $x \in V$ *and* $\lambda, \mu \in [0, 1]$.

Then $H(\cdot, 0)$ *has a fixed point if and only if* $H(\cdot, 1)$ *has a fixed point.*

3.2. **Application to the nonlinear Volterra integral equations.** In this part, we apply Theorem 2.2 to investigate the existence and uniqueness of a solution for the following nonlinear Volterra integral equation:

$$x(t) = \phi(t) + \int_a^t K(t, s, x(s))ds, \tag{3.23}$$

where $a, b \in \mathbb{R}$ with $a < b$, $x \in C[a, b]$ (the set of all continuous functions form $[a, b]$ into $\mathbb{R}$) is an unknown function, $\phi : [a, b] \to \mathbb{R}$ and $K : [a, b] \times [a, b] \times \mathbb{R} \to \mathbb{R}$ are two given continuous functions.

Theorem 3.4. *Consider the nonlinear Volterra equation (3.23). Suppose that there exists* $\lambda \in \Lambda$ *such that*

$$|K(t, s, r_1) - K(t, s, r_2)| \le \frac{\lambda(|r_1 - r_2|)}{b - a}$$

for all $t, s \in [a, b]$ *and* $r_1, r_2 \in \mathbb{R}$. *Then the nonlinear integral equation (3.23) has a unique solution.*

Proof. Let $X = C[a, b]$. Define the integral operator $T : X \to X$ for each $x \in X$ by a new function $Tx : [a, b] \to \mathbb{R}$ given by

$$(Tx)(t) = \phi(t) + \int_a^t K(t, s, x(s))ds$$

for all $t \in [a, b]$. We consider the complete metric space (X, d), where d is defined by

$$d(x, y) = \max_{t \in [a,b]} |x(t) - y(t)|$$

for each $x, y \in X$. Now, we define the functions $G : [0, \infty)^3 \to [0, \infty)$ and $\varphi : X \to [0, \infty)$ as follows:

$$G(a, b, c) = \max\{a, b\} + [c] \ \text{ for each } \ a, b, c \in [0, \infty),$$

$$\varphi(x) = 0 \ \text{ for each } \ x \in X.$$

Obviously, $G \in \mathcal{G}$ and $\lambda \in \Lambda$.

Next, we will show that (2.2) holds. Let $x, y \in X$ and $t \in [a, b]$. Then

$$\begin{aligned} |(Tx)(t) - (Ty)(t)| &= \left| \int_a^t K(t, s, x(s))ds - \int_a^t K(t, s, y(s))ds \right| \\ &= \left| \int_a^t (K(t, s, x(s)) - K(t, s, y(s)))ds \right| \\ &\leq \int_a^t |(K(t, s, x(s)) - K(t, s, y(s)))|ds \\ &\leq \frac{1}{b-a} \int_a^t \lambda(|x(s) - y(s)|)ds \\ &\leq \frac{1}{b-a} \int_a^t \lambda(d(x, y))ds \\ &\leq \frac{1}{b-a} \lambda(d(x, y))[b - a] \\ &= \lambda(d(x, y)). \end{aligned}$$

From the above inequality and by taking the maximum over t, we obtain

$$d(Tx, Ty) \leq \lambda(d(x, y)).$$

Consequently, it follows that

$$\max\{d(Tx, Ty), \varphi(Tx)\} + [\varphi(Ty)] \leq \lambda\big(\max\{d(x, y), \varphi(x)\} + [\varphi(y)]\big).$$

Hence,

$$G(d(Tx, Ty), \varphi(Tx), \varphi(Ty)) \leq \lambda\big(G(d(x, y), \varphi(x), \varphi(y))\big)$$

for all $x, y \in X$. Therefore, the inequality (2.2) together with all the assumptions of Theorem 2.2 are satisfied. Consequently, T has a unique fixed point, which implies that the integral equation (3.23) admits a unique solution in $C[a, b]$. □

4. Conclusion

In this work, we proved a φ-fixed point result for mappings satisfying the contractive condition involving control functions that do not have to be continuous and showed that our main results allowed us to find φ-fixed points of mappings in which the main results of Banach and Boyd-Wong cannot be applied, with Example 2.3. This claims the advantage of the main result of this work, with many known results from the past. Our results generalize the main φ-fixed point results of Karapinar et al. [7] and several known results in the literature. Actually, we can use the main φ-fixed point result in this paper to investigate the generalization of the fixed point result in the framework of partial metric spaces introduced by Karapinar et al. [7] (Corollary 3.3). This result will cover the partial metric version of the Boyd-Wong fixed point theorem and the fixed point result in partial metric spaces, as presented by Matthews [9].

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PATHAITHEP KUMROD
THAMMASAT UNIVERSITY
DEPARTMENT OF MATHEMATICS AND STATISTICS
PHAHOLYOTHIN ROAD, 12120, PATHUM THANI, THAILAND
Email address: phai.pathaithep@gmail.com

WUTIPHOL SINTUNAVARAT
THAMMASAT UNIVERSITY
DEPARTMENT OF MATHEMATICS AND STATISTICS
PHAHOLYOTHIN ROAD, 12120, PATHUM THANI, THAILAND
Email address: wutiphol@mathstat.sci.tu.ac.th