ALTAY CONFERENCE PROCEEDINGS IN MATHEMATICS
1 (2025), No. 1, pp. 54-67
https://altayconfproceedings.com/

Research Article

Distortion and quasisymmetric functions in quasiconformal mappings

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ABSTRACT. In this paper, we study the applications of special functions and quasisymmetry in quasiconformal mappings. Moreover, we estimate the distances between the image points of quasiconformal mappings under various metrics.

Keywords: Quasiconformal mapping, hyperbolic metric, distortion theorems.

2020 Mathematics Subject Classification: 30C65, 51M10.

1. INTRODUCTION

In the two-dimensional setting, the foundations of quasiconformal mapping theory were laid by H. Grötzsch in 1928, followed by significant contributions from O. Teichmüller during the period 1938–1944, and later by L. Bers and L. V. Ahlfors beginning in the early 1950s. The extension of this theory to higher dimensions (n-dimensional spaces) was initiated by F. W. Gehring and J. Väisälä in the early 1960s. Conformal invariants, such as the modulus of a family of curves, play a central role in the study of quasiconformal mappings. These invariants can often be characterized using specific conformal mappings. In the context of hyperbolic geometry on the unit disk, hyperbolic lines are defined as geodesicscurves that minimize length with respect to the hyperbolic metric. While this metric serves as a powerful tool for analyzing various planar domains, its extension to proper subdomains of higher-dimensional Euclidean spaces $\mathbb{R}^n$, with $n \geq 3$, is generally not feasible.

Classical Function Theory (CFT) extensively utilizes the hyperbolic metric due to its invariance under conformal mappings, particularly Möbius transformations. This invariance often renders theoretical results more naturally and effectively expressed in hyperbolic terms than in Euclidean geometry. A notable example is the Schwarz lemma, which characterizes analytic self-maps of the unit disk as contractions with respect to the hyperbolic metric. Similarly, Nevanlinnas principle underscores the fundamental role of hyperbolic geometry in the analytic structure of CFT. While classical tools such as power series expansions and integral representations are inherently local and do not capture global conformal invariance, the method of extremal lengthdeveloped by Ahlfors and Beurlingpreserves this invariance and has become a powerful and essential technique in modern CFT. However, generalizing these concepts to higher-dimensional settings ($n \geq 3$) poses significant difficulties, primarily due to the breakdown of complex analytic tools such as multiplication of complex numbers and the failure of

Received: 13.08.2025; Accepted: 06.10.2025; Published Online: 22.10.2025
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DOI: 10.64700/altay.4
Presented in *3rd International Conference: Constructive Mathematical Analysis*

the Riemann mapping theorem in higher dimensions. Liouvilles theorem further illustrates the rigidity of conformal mappings in higher dimensions, where they reduce to Möbius transformations, thus limiting the scope of conformal theory. Consequently, extending classical function theoretic methods to higher-dimensional spaces requires fundamentally different approaches and presents deep analytical challenges [26].

In this paper, we present a concise overview of quasiconformal mappings, various metric structures, and special functions, with a focus on their applications in deriving distortion theorems for quasiconformal mappings. The study is grounded in the foundational and recent developments presented in [3, 6, 7, 9, 10, 13, 17, 20, 24, 26, 22].

The structure of the paper is as follows. Section 1 provides an introduction to the topic. Section 2 presents the necessary definitions and explores the interrelations among various metric. In Section 3, we introduce certain special functions along with their associated inequalities, formulated as lemmas, and discuss the notion of quasi-symmetry. Section 4 offers a concise account of Moris theorem and the Schwarz lemma, including some of their recent advancements. Finally, Section 5 contains key lemmas and distortion theorems formulated with respect to different metrics.

2. Definitions and metrics

In our study, the following metric spaces will be central to our discussion:

(1) the Euclidean space $\mathbb{R}^n$,
(2) the Poincaré half-space $\mathbb{H}^n = \mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_n > 0\}$,
(3) the Möbius space $\overline{\mathbb{R}}^n = \mathbb{R}^n \cup \{\infty\}$.

2.1. **Notations.** For $x \in \mathbb{R}^n$ and $r > 0$ let

$$\mathbb{B}^n(x,r) = \{z \in \mathbb{R}^n : |x-z| < r\},$$
$$S^{n-1}(x,r) = \{z \in \mathbb{R}^n : |x-z| = r\}$$

denote the ball and sphere, respectively, centred at x with radius r. Moreover, we denote $\mathbb{B}^n(r) = \mathbb{B}^n(0,r)$, $S^{n-1}(r) = S^{n-1}(0,r)$, $\mathbb{B}^n(1) = \mathbb{B}^n$, $S^{n-1}(1) = S^{n-1}$. We denote by $e_1, e_2, \ldots, e_n$ the standard unit vectors in $\mathbb{R}^n$.

2.2. **Metric space** (X,d)**.** A metric on a non-empty set X is a real-valued function $d : X \times X \to [0,\infty)$ satisfying the following properties:

(1) $d(x,y) \geq 0$ and $d(x,x) = 0, \quad \forall x,y \in X$,
(2) $d(x,y) = d(y,x), \forall x,y \in X$,
(3) $d(x,z) \leq d(x,y) + d(y,z), \quad \forall x,y,z \in X$.

The metric space (X,d) is a set X equipped with a metric d on X. For instance,

(1) the set of real numbers $\mathbb{R}$, with the usual distance function $d(x,y) = |x-y|$, $x,y \in \mathbb{R}$ is a metric space.
(2) More generally $(\mathbb{R}^n, |\cdot|)$ is a metric space.
(3) If (X_j, d_j), $j = 1,2$ are metric spaces and $f : (X_1,d_1) \to (X_2,d_2)$ is an injection, then $m_f(x,y) = d_2(f(x),f(y))$ is a metric.
(4) If (X,d) is a metric space, then also (X,d^a) is a metric space for all $a \in (0,1]$.

2.3. **Isometry.** Let (X,d_1) and (Y,d_2) be metric space and let $f : X \to Y$ be homeomorphism. Then f is an isometry if

$$d_2\left(f(x),f(y)\right) = d_1(x,y)$$

for all $x,y \in X$.

2.4. **Lipschitz mappings.** Let (X, d_1) and (Y, d_2) be metric space and let $f : X \to Y$ be continuous and let $L \geq 1$. We say that f is L-Lipschitz if

$$d_2(f(x), f(y)) \leq L d_1(x, y)$$

for all $x, y \in X$. In addition, if f is homeomorphism and

$$d_1(x, y)/L \leq d_2(f(x), f(y)) \leq L d_1(x, y)$$

for all $x, y \in X$, we say that f is L-Lipschitz.

2.5. **Möbius transformation.** The following types of mappings generate the group of Möbius transformation.

(1) A reflection in hyperplane $P(a, t)$:

$$f_1(x) = x - 2(xa - t)\frac{a}{|a|^2},$$

where $P(a, t) = \{x \in \mathbb{R}^n : x \cdot a = t\}$, $t \in \mathbb{R}$, $a \in \mathbb{R}^n$.

(2) An inversion in $\mathbb{S}^{n-1}(a, r)$:

$$f_2(x) = a + \frac{r^2(x - a)}{|x - a|^2}, \quad f_2(a) = \infty,\ f_2(\infty) = a.$$

If $G \subset \overline{\mathbb{R}}^n$, we denote by $\mathcal{GM}(G)$ the group of all Möbius transformations with $f(G) = G$. For the further information and results on Möbius transformation the reader is referred to Beardon's book [5, p.32]. If f is an inversion in $S^{n-1}(a, r)$, then the following identity

$$|f(x) - f(y)| = \frac{r^2|x - y|}{|x - a||y - a|}$$

holds for $x, y \in \mathbb{R}^n \backslash \{a\}$, see [24, (1.5)].

Let $f(u) = r^2 u/|u|^2$, $r > 0$, $u \in \mathbb{R}^n \setminus \{0\}$ and let $z = x\left(|x| + |x - y|\right)/|x|$ for all $x, y \in \mathbb{R}^n \setminus \{0\}$ with $|x| \leq |y|$. Then

$$|f(x) - f(z)| \leq |f(x) - f(y)| \leq 3|f(x) - f(z)|.$$

Equality holds in the upper bound for $x = -y$ (see [7] for proof).

For $a \in \mathbb{B}^n$, define $T_a : \mathbb{B}^n \to \mathbb{B}^n$ as

$$T_a(x) = p_a \circ \sigma_a\,,$$

where p_a is the reflection in $(n-1)$-dimensional plane through the origin and orthogonal to a, and σ_a is the inversion in the sphere $S^{n-1}(a/|a|, \sqrt{1/|a|^2 - 1})$. For $n = 2$, we have

$$T_a(z) = \frac{z - a}{1 - \bar{a}z},$$

see [13, p.11, p.459].

2.6. **Absolute (cross) ratio.** For distinct points $a, b, c, d \in \overline{\mathbb{R}}^n$, the absolute (cross) ratio is defined by

$$|a, b, c, d| = \frac{q(a, c)q(b, d)}{q(a, b)q(c, d)},$$

where for all $x, y \in \mathbb{R}^n$

$$\begin{cases} q(x, y) = \frac{|x-y|}{\sqrt{1+|x|^2}\sqrt{1+|y|^2}}, \\ q(x, \infty) = \frac{1}{\sqrt{1+|x|^2}}, \end{cases}$$

is the spherical (chordal) metric [24]. The chordal metric is estimated as follows

$$\frac{|x-y|}{|1+(|x|)||(1+|y|)|} \leq q(x,y) \leq \frac{2|x-y|}{|1+(|x|)||(1+|y|)|},$$

for all $x, y \in \mathbb{R}^n$ [24, p.5]. If $q(f(x), f(y)) = q(x,y)$ for all $x, y \in \overline{\mathbb{R}}^n$ and $f \in \mathcal{GM}(\overline{\mathbb{R}}^n)$, then f is called spherical isometry. One of the most important properties of Möbius transformations is that they preserve absolute (cross) ratio, i.e.,

$$|f(a), f(b), f(c), f(d)| = |a, b, c, d|$$

for all $a, b, c, d \in \overline{\mathbb{R}}^n$ and $f \in \mathcal{GM}$.

2.7. **Conformal mapping.** Let D and D' be domains in $\mathbb{R}^n$ and $f : D \to D'$ be a homeomorphism. We call f *conformal* if $f \in \mathcal{C}^1$, $J_f(x) \neq 0$ for all $x \in D$, $|f'(x)h| = |f'(x)||h|$ for all $x \in D$ and $h \in \mathbb{R}^n$. If D and D' are domains in $\overline{\mathbb{R}}^n$, we call a homeomorphism $f : D \to D'$ conformal if the restriction of f to $D \setminus \{\infty, f^{-1}(\infty)\}$ is conformal.

For $n = 2$ there are many conformal mappings, i.e., Riemann mapping Theorem, Schwarz-Christoffel formula. For $n \geq 3$ conformal maps are Liouville's theorem, Möbius transformations. Therefore conformal invariance for the space $n \geq 3$ is very different from the plane case $n = 2$ [26].

2.8. **Quasiconformal mappings.** Given domains $D, D' \in \overline{\mathbb{R}}^n$. Let $f : D \to D'$ be a homeomorphism. If Γ is a family of curves in D, then $\Gamma' = f(\Gamma)$. We set

$$K_1(f) = \sup \frac{M(\Gamma')}{M(\Gamma)}, K_0(f) = \sup \frac{M(\Gamma)}{M(\Gamma')},$$

where the suprema is taken over is taken over all curve families Γ in D such that $M(\Gamma)$ and $M(\Gamma')$ are not simultaneously 0 or ∞. We call $K_1(f)$ the inner dilatation, $K_0(f)$ the outer dilatation, and $K(f) = \max\{K_1(f), K_0(f)\}$ the maximum dilatation of f. If $K(f) \leq K < \infty$, then f is known as K-quasiconformal. If $K(f) < \infty$, then f is quasiconformal. This geometric definition is due to Väisälä [22, p.42].

In particular, a 1-quasiconformal $f : \overline{\mathbb{R}}^n \to \overline{\mathbb{R}}^n$ is a Möbius transformation.

A set G is connected if for all $x, y \in G$ there exists a path $\gamma : [0,1] \to G$ such that $\gamma(0) = x$ and $\gamma(1) = y$. Sometimes, we write Γ_{xy} for the set of all paths joining x with y in G.

2.9. **Inner metric of a set.** Let $G \subset X$. For fixed $x, y \in X$ the inner metric with respect to G is defined by

$$d(x,y) = \inf\{\ell(\gamma) : \gamma \in \Gamma_{xy}, \gamma \in G\}.$$

2.10. **Geodesics.** A path $\gamma : [0,1] \to G$ where G is a domain, is a geodesic joining $\gamma(0)$ and $\gamma(1)$ if $\ell(\gamma) = d(\gamma(0), \gamma(1))$ and

$$\ell(\gamma) = d(\gamma(0), \gamma(t)) + d(\gamma(t) + \gamma(1)), \quad t \in (0,1).$$

For example, the segment

$$[x,y] = \{z \in \mathbb{R}^n : z = tx + (1-t)y, t \in [0,1]\}$$

is a geodesic in $(\mathbb{R}^n, |\cdot|)$.

2.11. **Path integrals.** For a locally rectifiable path $\gamma : \Delta \to X$ and a continuous function $f : \gamma\Delta \to [0,\infty]$, the path integral is defined in two steps as follows:

(1) If γ is rectifiable, then
$$\int_\gamma f ds = \int_0^{\ell(\gamma)} f(\gamma^o(t))|(\gamma^o)'(t)|dt,$$
where γ^o is the normal representation of a rectifiable path.

(2) If γ is locally rectifiable, then we set
$$\int_\gamma f ds = \sup\left\{\int_\alpha \,:\, \ell(\alpha) < \infty,\, \alpha \,\text{is a subpath of }\, \gamma)\right\}.$$

2.12. **Weighted length.** Let $G \subset X$ be a domain and $w : G \to (0,\infty)$ continuous. For fixed $x, y \in D$, we define
$$d_w(x,y) = \inf\left\{\ell_w(\gamma) = \int_\gamma w(\gamma(z))|dz| : \gamma \in \Gamma_{xy},\, \ell(\gamma) < \infty\right\}.$$
One can see easily that d_w defines a metric on G and (G, d_w) is a metric space.

2.13. **Hyperbolic metric.** Let Γ be a family of all rectifiable curves in $\mathbb{B}^n$ or $\mathbb{H}^n$, joining x and y. Then the hyperbolic metric ρ in unit ball $\mathbb{B}^n$ and in $\mathbb{H}^n$ is defined by
$$\rho_{\mathbb{B}^n} = \inf_{\gamma\in\Gamma}\int_\gamma \frac{2|dz|}{1-|z|^2}, \quad x,y \in \mathbb{B}^n \quad \text{and} \quad \rho_{\mathbb{H}^n} = \inf_{\gamma\in\Gamma}\int_\gamma \frac{|dz|}{d(z,\partial\mathbb{H}^n)}, \quad x,y \in \mathbb{H}^n,$$
respectively.

If D is a simply connected domain of the extended complex plane $\mathbb{C}$ and card$\partial D > 1$, then by the Riemann mapping theorem there exists a conformal mapping f from D onto the unit disk $\{z \in C : |z| < 1\}$.

The hyperbolic metric ρ in D is defined by the following formula
$$\rho_D(x,y) = \inf_{\gamma\in\Gamma}\int_\gamma \eta_D(z)|dz|, \quad \text{where} \quad \eta_D(z) = \frac{2|f'(z)|}{1-|f(z)|^2},$$
where Γ is the family of all rectifiable curves in D joining x and y. The following inequalities give us an upper and lower bound for η_D
$$\frac{1}{4d(z,\partial D)} \le \eta_D(z) \le \frac{1}{d(z,\partial D)}.$$
Once the Euclidean distances $x_n = d(x,\partial\mathbb{H}^n)$, $y_n = d(x,\partial\mathbb{H}^n)$ and $|x-y|$ are known then the hyperbolic distance $\rho_{\mathbb{H}^n}$ can be determined by the following formula
$$\cosh(\rho(x,y)) = 1 + \frac{|x-y|^2}{2x_n y_n}, \quad x,y \in \mathbb{H}^n.$$
The counterpart of the above formula for $\mathbb{B}^n$ is given by
$$\sinh^2\left(\frac{1}{2}\rho(x,y)\right) = \frac{|x-y|^2}{(1-|x|^2)(1-|y|^2)}, \quad x,y \in \mathbb{B}^n,$$
see [5, p.35-40]. For the unit ball $\mathbb{B}^n$, we have two more definition for the hyperbolic metric given below
$$\rho_{\mathbb{B}^n}(x,y) = \sup\{\log|a,x,y,b|\}, \quad x,y \in \mathbb{B}^n,\ a,b \in \partial\mathbb{B}^n,$$
$$\rho_{\mathbb{B}^n}(x,y) = \log|x_*,x,y,y_*|, \quad x,y \in \mathbb{B}^n\,.$$

If $h \in \mathcal{GM}$ and $x, y \in \mathbb{B}^n$, then

$$\rho(h(x), h(y)) = \rho(x, y), \quad x, y \in \mathbb{B}^n.$$

2.14. **Metrics j and $\tilde{j}$.** Let (X, d) be a metric space and $G \subset X$ be an open set with non-empty boundary. Then for all $x, y \in G$, the following formulas

$$j_G(x, y) = \log\left(1 + \frac{|x-y|}{\min\{d(x), d(y)\}}\right)$$

and

$$\tilde{j}_G(x, y) = \log\left(1 + \frac{|x-y|}{d(x)}\right) + \log\left(1 + \frac{|x-y|}{d(y)}\right)$$

define metrics on G (see [12, 24]), where $d(x) = d(x, \partial G) = \inf\{|x - z| : z \in \partial G\}$ is the distance between a point $x \in G$ and the boundary ∂G of G. In literature, metrics j and $\tilde{j}$ are known as distance ratio metric and ratio metric, respectively.

The following inequalities show the relation between $j, \tilde{j}$ and ρ metrics in different domains [24, p.29],

(1)
$$j_G(x, y) \le \tilde{j}_G(x, y) \le 2j_G(x, y) \quad \text{for all } x, y \in G \subset \mathbb{R}^n,$$
(2)
$$j_{\mathbb{B}^n}(x, y) \le \rho_{\mathbb{B}^n}(x, y) \le 4j_{\mathbb{B}^n}(x, y) \quad \text{for all } x, y \in \mathbb{B}^n,$$
(3)
$$j_{\mathbb{H}^n}(x, y) \le \rho_{\mathbb{H}^n}(x, y) \le 2j_{\mathbb{H}^n}(x, y) \quad \text{for all } x, y \in \mathbb{H}^n.$$

2.15. **Quasihyperbolic metric.** Let $G \subset \mathbb{R}^n$ be a domain. The quasihyperbolic metric k_G is defined by

$$k_G(x, y) = \inf_{\gamma \in \Gamma} \int_\gamma \frac{|dz|}{d(z, \partial G)}, \quad x, y \in G,$$

where Γ is the family of all rectifiable curves in G joining x and y [12]. In the following inequalities, we show the relation of k_G metric with other metrics [12, 16, 24, 26],

(1)
$$\frac{1}{2}\rho_{\mathbb{B}^n}(x, y) \le k_{\mathbb{B}^n}(x, y) \le \rho_{\mathbb{B}^n}(x, y) \quad \text{for all } x, y \in \mathbb{B}^n,$$
(2)
$$j_{\mathbb{B}^n}(x, y) \le k_{\mathbb{B}^n}(x, y) \le (1 + r)j_{\mathbb{B}^n}(x, y) \quad \text{for all } x, y \in \mathbb{B}^n(r),\ r \in (0, 1),$$
(3)
$$j_G(x, y) \le \log\left(1 + \frac{l}{\min\{d(x), d(y)\}}\right) \le k_G(x, y) \quad \text{for all } x, y \in G \subset \mathbb{R}^n,$$
where $l = \inf\{\ell(\gamma) : \gamma \in \Gamma_{x,y}\}$,

(4)
$$j_{\mathbb{B}^n}(0, y) = k_{\mathbb{B}^n}(0, y) = \log\frac{1}{1 - |x|} \quad \text{for all } x \in \mathbb{B}^n,$$
(5)
$$j_{\mathbb{B}^n}(ax, ay) = k_{\mathbb{B}^n}(ax, ay) = \frac{1-x}{1-y}, \quad a \in S^{n-1},$$
where $x, y \in (0, 1)$ with $x < y$.

2.16. **Ferrand's metric.** Let $D \subset \overline{\mathbb{R}}^n$ be a domain with card$\partial D \geq 2$. Then τ_D defines a metric in D, defined

$$\sigma_D(x,y) = \inf_{\gamma\in\Gamma} \int_\gamma w_D(z)|dz|,$$

where Γ is the family of all rectifiable curves in D joining x and y, and

$$w_D = \sup_{a,b\in\partial D} \frac{|a-b|}{|z-a||z-b|}.$$

2.17. **Apollonian metric.** For a proper subdomain D of $\overline{\mathbb{R}}^n$, the Apollonian metric is defined by

$$\alpha_D(x,y) = \sup_{a,b\in\partial D} \log|a,x,y,b|, \quad x,y \in D.$$

2.18. **Absolute ratio metric.** Let G be an open subset of $\overline{\mathbb{R}}^n$ with card$\partial G \geq 2$. Then for all $x, y \in G$ the following formula

$$\delta_G(x,y) = \log\left(1 + \sup_{a,b\in\partial G} |a,x,y,b|\right),$$

defines a metric in G ([20]).

3. Special functions

3.1. **Modulus of curve family.** Let Γ be a family of curves in $\overline{\mathbb{R}}^n$. We denote by $F(\Gamma)$ the set of all non-negative Borel functions $\rho : \overline{\mathbb{R}}^n \to \mathbb{R}^n \cup \{\infty\}$ such that $\int_\gamma \rho ds \geq 1$ for every locally rectifiable curves $\gamma \in \Gamma$. We define the modulus

$$M(\Gamma) = \inf_{\rho\in F(\Gamma)} \int_{\mathbb{R}^n} \rho^n dm,$$

where m stands for the n-dimensional Lebesgue measure.

For sets $E, F \subset G$, and $G \subset \overline{\mathbb{R}}^n$, we denote by $\Delta(E,F;G)$ the curve family of all curves joining the sets E and F in G, and let $\Delta(E,F) = \Delta(E,F;\overline{\mathbb{R}}^n)$. The capacity of a ring $R(E,F)$ is cap$R(E,F) = M(\Delta(E,F))$, where M is modulus of family of curves $\Delta(E,F)$.

3.2. **Conformal invariants.** Let G be a proper subdomain of $\overline{\mathbb{R}}^n$. For $x, y \in G$ with $x \neq y$ we define

$$\lambda_G(x,y) = \inf_{C_x,C_y} M(\Delta(C_x,C_y;G)),$$

where $C_z = \gamma_z[0,1]$ and $\gamma_z : [0,1) \to G$ is a curve such that $\gamma_z(0) = z$ and $\gamma_z(t) \to \partial G$ when $t \to 1$, $z = x, y$.

If G is a proper subdomain of $\overline{\mathbb{R}}^n$. Then for $x, y \in G$, we define

$$\mu_G(x,y) = \inf_{C_{xy}} M(\Delta(C_{xy}, \partial G; G),$$

where the infimum is taken over all continua C_{xy} such that $C_{xy} = \gamma[0,1]$ and γ is a curve with $\gamma(0) = x$ and $\gamma(1) = y$.

Conformal invariant λ_G was introduced by Ferrand [25]. Teichmüller studied λ_G for $n = 2$ and $G = \mathbb{R}^2 \setminus \{0\}$, while μ_G was studied by Grötzsch for $n = 2$ and $G = \mathbb{B}^2$, see [1]. It was remarked in [3, p.320] that $\lambda_G^{1/(1-n)}$ and μ_G are metrics.

For all $x, y \in D$ with $x \neq y$ and $d \in \{\mathbb{H}^n, \mathbb{B}^n\}$, the following relations hold true [14]

$$\lambda_D(x,y) = 2^{-n}\tau_n(\cosh(\rho_D(x,y)/2)),$$

$$\mu_D(x,y) = \tau_n(1/\tanh(\rho_D(x,y)/2)).$$

It was proved in [14] that metrics μ_{H^2}, μ_{B^2} and $\lambda_{B^2}^{-1}$ are not Hölder continuous with respect to the hyperbolic metric.

3.3. **Rings.** A domain G in $\overline{\mathbb{R}}^n$ is called a *ring* if $\overline{\mathbb{R}}^n \setminus G$ has two components. If the components are C_0 and C_1 we write $G = R(C_0, C_1)$. For $s > 1$, the complementary components of the Grötzsch ring $R_{G,n}$ in $\mathbb{R}^n$ are $\overline{\mathbb{B}}^n$ and $[se_1, \infty]$. For $t > 0$, the complementary components of the Teichmüller ring $R_{T,n}$ in $\mathbb{R}^n$ are $[-e_1, 0]$ and $[te_1, \infty]$.

The Grötzsch ring constant λ_n is defined by

$$\log \lambda_n = \lim_{r \to 0+} (M_n(r) + \log r),$$

where $M_n(r)$ is the conformal modulus of the Grötzsch ring [3, p.167].

The capacities of $T_{G,n}$ and $R_{G,n}(s)$ are denoted by decreasing homeomorphism functions $\tau_n : (0,\infty) \to (0,\infty)$ and $\gamma_n : (1,\infty) \to (0,\infty)$, respectively, with the following formula [13, p.121]

$$\tau_n(s) = M(\Delta([-e_1,0],[se_1,\infty];\mathbb{R}^n)), \quad s > 0,$$
$$\gamma_n(s) = M(\Delta(\overline{\mathbb{B}}^n,[se_1,\infty];\mathbb{R}^n), \quad s > 1.$$

Moreover, the functions τ_n and γ_n satisfying the following functional identity

$$\gamma_n(t) = 2^{n-1}\tau_n(t^2-1), \quad t > 1,$$

[24, Lemma 5.53].

Utilizing the capacities τ_n and γ_n of $T_{G,n}$ and $R_{G,n}(s)$, respectively, the following identities were proved in [24, Thm 8.6],

$$\mu_{\mathbb{B}^n}(x,y) = \gamma_n\left(\frac{1}{\tanh(\rho(x,y)/2)}\right),$$
$$\lambda_{\mathbb{B}^n}(x,y) = \frac{\tau_n}{2}\left(\tanh^2(\rho(x,y)/2)\right)$$

for all $x, y \in \mathbb{B}^n$.

3.4. **Elliptic integrals.** The plane Grötzsch ring can be mapped onto an annulus by an elliptic function [5]

$$\tau_2(s) = \frac{2\pi}{\mu(1/s)}, \quad s > 1,$$

where

$$\mu(r) = \frac{\pi}{2}\frac{K\sqrt{1-r^2}}{\mathcal{K}(r)}, \quad \mathcal{K}(r) = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-r^2t^2)}}.$$

The function $\mathcal{K}(r)$ is called a complete elliptic integral of the first kind. The function $\mu(r)$ satisfies the following functional identities

$$\mu(r) = 2\mu\left(\frac{2\sqrt{r}}{1+r}\right), \quad \mu(r)\mu\left(\frac{1-r}{1+r}\right) = \frac{\pi^2}{2}, \quad \mu(r)\mu(\sqrt{1-r^2}) = \frac{\pi^2}{4}.$$

For $0 < r < 1$, the function $\mu(r)$ is estimated as below [24, p.67]

$$\log\left(\frac{1}{r}\right) < \mu(r) < \log\left(\frac{4}{r}\right).$$

3.5. **Distortion function.** For all $K>0$ and $n\geq 2$, the distortion function $\varphi_{K,n}:[0,1]\to[0,1]$ is defined by

$$\varphi_{K,n}(r)=\frac{1}{\gamma_n^{-1}(K\gamma_n(1/t))},\quad 0<t<1,$$

$\varphi_{K,n}(0)=0$ and $\varphi_{K,n}(1)=1$.

For later use we fix the constants α and β as follows

$$\alpha=K^{1/(1-n)}=\beta,\quad K\geq 1\,.$$

Lemma 3.1. *For $n\geq 2$, $K\geq 1$, and $r\in[0,1]$, we have*

$$\left\{\varphi_{K,n}(r)\leq\lambda_n^{1-\alpha}r^{\alpha},\quad \varphi_{1/K,n}(r)\geq\lambda_n^{1-\beta}r^{\beta}\right. \tag{3.1}$$

and

$$\left\{\lambda_n^{1-\alpha}\leq 2^{1-\alpha}K\leq 2^{1-1/K}K,\quad \lambda_n^{1-\beta}\geq 2^{1-\beta}K^{-\beta}\geq 2^{1-K}K^{-K}\right. \tag{3.2}$$

see [24, Thm 7.47] *and* [3, Lem 875].

For $n\geq 2$, $t\in(0,\infty),\ K>0$, we denote

$$\eta_{K,n}(t)=\tau_n^{-1}\left(\frac{1}{K}\tau_n(t)\right)=\frac{1-\varphi_{1/K,n}(1/\sqrt{1+t})^2}{\varphi_{1/K,n}(1/\sqrt{1+t})^2}\,.$$

For $n\geq 2$, $1\leq K<\infty, t\in[0,\infty)$, let

$$\begin{aligned}\eta^*_{K,n}(t)=\sup\{&|f(x)|:|x|\leq t,\ f\in QC_K(\overline{\mathbb{R}}^n),\\ &f(0)=0,\ f(e_1)=e_1,\ f(\infty)=\infty\},\end{aligned}$$

where $QC_K(\mathbb{B}^n)$ denotes the set of all K-quasiconformal maps of $\overline{\mathbb{R}}^n$ into itself. It is well know that $\eta^*_{K,2}(t)=\eta_{K,2}(t)$, [17, p.80].

For $n\geq 2$ and $0<r<1$,

$$\varphi^*_{K,n}(r)=\begin{cases}\sup\{|f(x)|:|x|=r,f\in QC_K(\mathbb{B}^n),f(0)=0\} & \text{if } 1\leq K<\infty\\ \inf\{|f(x)|:|x|=r,f\in QC_{1/K}(\mathbb{B}^n),f(0)=0\} & \text{if } 0<K\leq 1.\end{cases}$$

Lemma 3.2. *For $n\geq 2$, $K\geq 1$, and $alpha=K^{1/(1-n)}=1/\beta$, $0<r<1$, we have* [3, 13(3)]

$$r^{\alpha}\leq\varphi^*_{K,n}(r)\leq\varphi_{K,n}(r)\leq\left(\frac{\lambda_n}{2}\right)^{1-\alpha}(1+r')^{1-\alpha}r^{\alpha}\leq\lambda_n^{1-\alpha}r^{\alpha}\,.$$

Lemma 3.3. *In the following inequalities we have upper bounds for $\eta^*_{K,n}$* [3, Thm 14.6 & 14.8]. *For $n\geq 2$ and $K\geq 1$, we have*

$$\begin{cases}\eta^*_{K,n}(t)\leq\eta^*_{K,n}(1)\varphi^*_{K,n}(t), & 0\leq t\leq 1,\\ \eta^*_{K,n}(t)\leq\eta^*_{K,n}(1)\varphi^*_{1/K,n}(t), & t\geq 1,\\ \eta^*_{K,n}(1)\leq e^{4K(K+1)\sqrt{K-1}}\,. & \end{cases}$$

3.6. **Quasisymmetic function.** Let $\eta:[0,\infty)\to[0,\infty)$ be a homeomorphism with $\eta(0)=0$. A homeomorphism between two metric spaces X and Y with distances denoted by $|x-y|$ is called η-quasisymmetric if

$$\frac{|f(x)-f(y)|}{|f(x)-f(z)|}\leq\eta\left(\frac{|x-y|}{|x-z|}\right)$$

for all distinct points $x,y,z\in D$ with $x\neq z$ (see [21]).

The quasisymmetry can be extended to maps $\overline{\mathbb{R}}^n$ by using absolute (cross) ratio

$$|f(a), f(b), f(c), f(d)| \leq \eta(|a, b, c, d|)$$

for all distinct points $a, b, c, d \in \overline{\mathbb{R}}^n$. Mappings satisfying this condition are called quasi-möbius mappings [23]. A K-quasiconformal mapping $f : \mathbb{R}^2 \to \mathbb{R}^2$ is $\eta^*_{K,2}$-quasisymmetric [3, Thm 9.42].

4. Mori's theorem and Schwarz Lemma

In 1956, A. Mori [18] proved the following theorem.

Theorem 4.1. *Let $f : \mathbb{B}^2 \to \mathbb{B}^2$ be a K-quasiconformal mapping with $f(0) = 0$ and $f(\mathbb{B}^2) = \mathbb{B}^2$. Then for all $x, y \in \mathbb{B}^2$*

$$|f(x) - f(y)| \leq 16|x - y|^K .$$

The number 16 cannot be replace by an smaller absolute constant.

In [17, p.68], it was conjectured that the best constant on place of 16 is $16^{1-1/K}$. In [3, Thm 15.4], it is proved that

$$|f(x) - f(y)| \leq 64^{1-1/K}|x - y|^K ,$$

where f is as in Theorem 4.1.

In 1997, Qiu [19] prove the best constant $46^{1-1/K}$ in place of 16. For $n \geq 2$, Fehlmann and Vuorinen [9] proved the following theorem.

Theorem 4.2. *Let $f : \mathbb{B}^n \to \mathbb{B}^n$ be a K-quasiconformal mapping with $f(0) = 0$ and $f(\mathbb{B}^n) = \mathbb{B}^n$. Then for all $x, y \in \mathbb{B}^n$*

$$|f(x) - f(y)| \leq M(n, K)|x - y|^\alpha , \quad \alpha = K^{1/(1-n)},$$

where the constant $M(n, K)$ has the following properties:

(1) $M(n, K) \to 1$ as $K \to 1$, uniformly in n,
(2) $M(n, K)$ remains bounded for fixed K and varying n,
(3) $M(n, K) \leq 3\lambda_n^2$ for all $K \geq 1$.

With the same assumption as in Theorem 4.2, Anderson and Vamanamurthy [2] proved that $M(n, K) \leq 4\lambda_n^{1(1-\alpha)}$, where $\alpha = K^{1/(1-n)}$, and $\lambda \in [4, 2e^{n-1})$ is the Grötzsch ring constant [24, p.89].

In 2011, Bhayo and Vuorinen [6] proved that for $n \geq 2,\ K \geq 1$,

$$M(n, K) \leq T(n,\ K) \leq \inf\{h(t) : t \geq 1\},$$

where $\alpha = K^{1/(1-n)} = 1/\beta$. There exists a number $K_1 > 1$ such that for all $K \in (1, K_1)$ the function h has a minumum at a point $t_1 > 1$ and

$$T(n, K) \leq h(t_1) = \frac{3^{1-\alpha}(\beta - \alpha)^{\alpha^2}}{\alpha^\alpha}\lambda_n^{\alpha-\alpha^2} + \lambda_n^{\beta-1}\left(\frac{3\alpha\lambda_n^{\alpha-1}}{(\beta - \alpha)^\alpha}\right)^{\eta-\alpha} .$$

Moreover, for $\beta \in (1, 2)$, we have

$$h(t_1) = 3^{\beta-\alpha}2^{1-\alpha}K^5\left(\frac{3}{2}\sqrt[4]{\beta - \alpha} + \exp(\beta - 1)\right).$$

In particular, $h(t_1) \to 1$ when $K \to 1$. For $K \in (1, 1.3044)$ and $K > 8.9105$ the bound $T(n, K)$ improves the bound $M(n, K)$ given in [2].

A counterpart of Theorem 4.2 was given as follows with some restrictions.

Theorem 4.3. *For $n \geq 2$, $K \geq 1$, let $f : \overline{\mathbb{R}}^n \to \overline{\mathbb{R}}^n$ be a K-quasiconformal mapping with $f(0) = 0$, $f(e_1) = e_1$, and $f(\infty) = \infty$. Then*

$$|f(x) - f(y)| \leq c(K)^2 |x-y|^\beta$$

for all $x \in \mathbb{R}^n \setminus \mathbb{B}^n$, $y \in \mathbb{R}^n$ with $1 < |x| < |x-y|$, and

$$|f(x) - f(y)| \leq c(K)^2 |x-y|^\alpha$$

for all $x \in \mathbb{B}^n$, $y \in \mathbb{R}^n$ with $|x-y| < |x| < 1$, where $c(K) = 2^{K-1} K^K e^{4K(K+1)\sqrt{K-1}}$.

Theorem 4.4 ([24, 11.2]). *Let $f : \mathbb{B}^n \to \mathbb{R}^n$ be a nonconstant K-quasiregular mapping* [22] *with $f(\mathbb{B}^n) \subset \mathbb{B}^n$ and $\alpha = K^{1/(1-n)}$. Then for all $x, y \in \mathbb{B}^n$*

$$\tanh\left(\frac{\rho(f(x), f(y))}{2}\right) \leq \varphi_{K,n}\left(\tanh\frac{\rho(x,y)}{2}\right) \leq \lambda_n^{1-\alpha}\left(\tanh\frac{\rho(x,y)}{2}\right).$$

If $f(0) = 0$, then for all $x \in \mathbb{B}^n$

$$|f(x)| \leq \lambda_n^{1-\alpha} |x|^\alpha .$$

For the plane case $n = 2$, the quasiconformal version of the above theorem appears in [17, p.65]. The following formula

$$\rho(f(x), f(x)) \leq K(\rho(x,y) + \log 4)$$

was proved in [8, Thm 5.1], where f is as in Theorem 4.4.

Under the same assumption as in Theorem 4.4, if $f(0) = 0$ then for all $x \in \mathbb{B}^n$

$$\varphi_{1/K,n}(|x|) \leq |f(x)| \leq \varphi_{K,n}(|x|),$$

[6, Lem 6.1].

For $n \geq 2$, $\alpha = K^{1/(1-n)}$, let $f : \overline{\mathbb{R}}^n \to \overline{\mathbb{R}}^n$ be a K-quasiconforma mapping fixing $0, e_1$, and ∞, then

$$|f(x)| \leq K \lambda_n^{2(1-\alpha)} |x|^\alpha$$

for all $x \in \mathbb{R}^n \setminus \{0\}$,$\mathbb{B}^n$ with $0 < |x| \leq (K-1)/K$.

Theorem 4.5 ([6]). *Let $f : \mathbb{B}^2 \to \mathbb{R}^2$ be a nonconstant K-quasiregular mapping* [22] *with $f(\mathbb{B}^2) \subset \mathbb{B}^2$, then for all $x, y \in \mathbb{B}^2$*

$$\rho(f(x), f(y)) \leq c(K) \max\{\rho(x,y), \rho(x,y)^{1/K}\},$$

where $c(K) = 2\mathrm{arctanh}(\varphi(\tanh(1/2)))$.

5. Quadruples and distortion theorems

Lemma 5.4 ([3, Thm 14.7]). *For $K \geq 1$ let f be a K-quaisconformal automorphism of the plane $\overline{\mathbb{R}}^2$. Then*

$$\frac{1}{\lambda(K)} \min\{t^{1/K}, t^K\} \leq |f(a), f(b), f(c), f(d)| \leq \lambda(K) \max\{t^{1/K}, t^K\},$$

for each ordered quadruple of distinct points a, b, c, d in the plane, where $t = |a, b, c, d|$. Moreover the inequalities are sharp for each $K \geq 1$.

Lemma 5.5. *For $K \geq 1$ let $f : \overline{\mathbb{R}}^2 \to \overline{\mathbb{R}}^2$ be K-quasiconformal. Then*

$$|f(a), f(b), f(c), f(d)| \leq e^{\pi K(K-1)} \max\{t^{1/K}, t^K\},$$

for each ordered quadruple of distinct points $a, b, c, d \in \mathbb{R}^2$.

Lemma 5.6 ([3, Thm 14.8]). *Let $f : \overline{\mathbb{R}}^n \to \overline{\mathbb{R}}^n$ be K-quasiconformal. Then*
$$\frac{1}{\eta^*_{K,n}}(1/|a,b,c,d|) \le |f(a),f(b),f(c),f(d)| \le \eta^*_{K,n}(|a,b,c,d|)\,,$$
for each ordered quadruple of distinct points $a,b,c,d \in \overline{\mathbb{R}}^n$.

Theorem 5.6 ([3, Thm 15.20]). *Let $G \subset \mathbb{R}^n$ and let $f : G \to G' \subset \overline{\mathbb{R}}^n$ be a K-quasiconformal. Then*
$$q(f(x),f(y))q(\partial G') \le 128\left(\frac{|x-y|}{d(x,\partial G)}\right)^{1/K},$$
where q is a chordal metric.

Theorem 5.7. *Let $G \subset \mathbb{R}^n$, $f : \overline{\mathbb{R}}^n \to \overline{\mathbb{R}}^n$ be K-quasiconformal. Then*
$$\delta_{fG}(f(x),f(y)) \le c(K)\beta\max\{\delta_G(x,y),\delta_G(x,y)^\alpha\},$$
where $c(K) = 2^{K-1}K^Ke^{4(K+1)\sqrt{K-1}}$ and $\alpha = K^{1/(1-n)} = 1/\beta$.

Theorem 5.8 ([20]). *Let $f : \overline{\mathbb{R}}^n \to \overline{\mathbb{R}}^n$ be a K-quasiconformal, G and fG are open sets of $\overline{\mathbb{R}}^n$ with card $\partial G \ge 2$, then*
$$\delta_{fG}(f(x),f(y)) \le \eta_{K,n}(\delta_G(x,y))$$
for all $x,y \in G$.

Theorem 5.9 ([20]). *Let $f : G \to G'$ be a K-quasiconformal mapping, where G and $fG = G'$ are proper subsets of $\overline{\mathbb{R}}^n$, then*
$$\frac{\mu_G(x,y)}{K} \le \mu'_G(f(x),f(y)) \le K\mu_G(x,y)$$
for all $x,y \in G$. If card $\partial G \ge 2$, *then*
$$\frac{\lambda_G(x,y)}{K} \le \lambda'_G(f(x),f(y)) \le K\lambda_G(x,y)$$
for all $x,y \in G$.

Theorem 5.10. *Let $G \subset \mathbb{R}^2$ and $f : \overline{\mathbb{R}}^2 \to \overline{\mathbb{R}}^2$ be a K-quasiconformal mapping with $f(\infty) = \infty$, then*
$$\tilde{j}_{fG}(f(z_1),f(z_2)) \le e^{\pi(K-1/K)}K\max\{\tilde{j}_G(x,y),\tilde{j}_G(x,y)\}$$
for all $z_1,z_2 \in G$.

If G is a disk or half plane then $\rho_G(z_1,z_2) \le \tilde{j}_G(z_1,z_2)$, for all $z_1,z_2 \in G$ ([11]).

Theorem 5.11 ([15]). *Let $G = \mathbb{R}^n \setminus \{0\}$, and $f : \overline{\mathbb{R}}^n \to \overline{\mathbb{R}}^n$ be a K-quasiconformal mapping with $f(0) = 0$ and $f(\infty) = \infty$, then*
$$j_{fG}(f(x),f(y)) \le a(K)\max\{j_G(x,y),j_G(x,y)\}$$
for all $x,y \in G$, where $\alpha = K^{1/(1-n)} = 1/\beta$, $a(K) = e^{60\sqrt{K-1}}$, and $a(K) \to 1$ as $K \to 1$.

The above theorem is refined as follows:

Theorem 5.12. *Let $G = \mathbb{R}^n \setminus \{0\}$, and $f : \overline{\mathbb{R}}^n \to \overline{\mathbb{R}}^n$ be a K-quasiconformal mapping with $f(0) = 0$ and $f(\infty) = \infty$, then*
$$j_{fG}(f(x),f(y)) \le b(K)\max\{j_G(x,y),j_G(x,y)\}$$
for all $x,y \in G$, where $\alpha = K^{1/(1-n)} = 1/\beta$, $b(K) = 2^{K-1}K^Ke^{(4K(K+1))\sqrt{K-1}}$, and $a(K) \to 1$ as $K \to 1$.

In [7] a two exponent variant of the function $x \mapsto |x|^{p-1}x$ was defined for $a, b > 0, x \in \mathbb{R}^n$,

$$\mathcal{A}_{a,b}(x) = \begin{cases} |x|^{a-1}x & if\ |x| < 1 \\ |x|^{b-1}x & if\ |x| \geq 1. \end{cases}$$

For $a = b$, the function $\mathcal{A}_{a,b}$ defines a quasiconformal mapping and it has been used in many examples to illuminate various properties of these maps [22, p.49]. For instance, if $a \in (0,1)$ the function $\mathcal{A}_{a,b}$ is Hölder-continuous at the origin.

The next theorem tells us how the hyperbolic distances from the origin are changed under the radial selfmapping of the the unit disk, $z \mapsto |z|^{1/K-1}z, K > 1$, which is the restriction of $\mathcal{A}_{1/K,1/K}(z)$ to the unit disk, see [6].

Theorem 5.13. *The following inequality holds for $K \geq 1$, $|z| < 1$;*

$$\rho(0, \mathcal{A}_{1/K,K}(z)) \leq K \max\{\rho(0,|z|), \rho^{1/K}(0,|z|)\},$$

where ρ is the hyperbolic metric [24, p. 19].

Corollary 5.1 ([7])**.**

(1) Let $D = \mathbb{R}^n \setminus \{0\}$, then we have

$$j_D(\mathcal{A}_{1/K,K}(x), \mathcal{A}_{1/K,K}(y)) \leq 2^{1-1/K} \max\{j_D(x,y), j_D^{1/K}(x,y)\}$$

for all $K \geq 1, x, y \in \mathbb{B}^n \cap D$.

(2) The following inequality holds for $K \geq 1$:

$$||x|^{K-1}x - |y|^{K-1}y| \leq e^{\pi(K-1/K)}|x|^{K-1/K} \max\{|x-y|^{1/K}, |x-y|^K\}$$

for all $x, y \in \mathbb{C} \setminus \overline{\mathbb{B}}^2$.

(3) The following inequality holds for $K \geq 1$ and for all $x, y \in \mathbb{R}^n \setminus \overline{\mathbb{B}}^n$:

$$||x|^{\beta-1}x - |y|^{\beta-1}y| \leq c(K)|x|^{\beta-\alpha} \max\{|x-y|^\alpha, |x-y|^\beta\},$$

where $c(K) = 2^{K-1}K^K \exp(4K(K+1)\sqrt{K-1})$ and $\alpha = K^{1/(1-n)} = 1/\beta$.

Corollary 5.2 ([7])**.** *The following inequalities hold for $K \geq 1$;*

$$\left|\frac{x}{|x|^{1+1/K}} - \frac{y}{|y|^{1+1/K}}\right| \leq 2^{1-1/K} \frac{|x-y|^{1/K}}{(|x||y|)^{1/K}} \tag{5.3}$$

for all $x, y \in \mathbb{R}^n \setminus \mathbb{B}^n$,

$$\left|\frac{x}{|x|^{1+\beta}} - \frac{y}{|y|^{1+\beta}}\right| \leq \frac{c(K)}{|x|^{\beta-\alpha}} \max\left\{\left(\frac{|x-y|}{|x||y|}\right)^\alpha, \left(\frac{|x-y|}{|x||y|}\right)^\beta\right\} \tag{5.4}$$

for all $x, y \in \mathbb{B}^n$,

$$\left|\frac{x}{|x|^{1+K}} - \frac{y}{|y|^{1+K}}\right| \leq \frac{e^{\pi(K-1/K)}}{|x|^{K-1/K}} \max\left\{\left(\frac{|x-y|}{|x||y|}\right)^{1/K}, \left(\frac{|x-y|}{|x||y|}\right)^K\right\} \tag{5.5}$$

for all $x, y \in \mathbb{B}^2$.

The proofs of the theorems presented in this section are primarily based on the definition of quasisymmetry (see 3.6) and Lemmas 3.1, 3.2, 3.3, 5.4, 5.5, 5.6. For details, the reader is referred to [7, 6, 4, 20, 3].

Acknowledgments

This work was carried out during the author's research visit to Leibniz University Hannover. The author gratefully acknowledges the financial support provided by the LUT Research Foundation.

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