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Research Article

Mono-unary condition for algebras with easy direct limits

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ABSTRACT. Let $\mathcal{A}$ be an algebra such that exactly algebras isomorphic to a retract of $\mathcal{A}$ can be constructed from $\mathcal{A}$ by direct limits. One condition which is satisfied for $\mathcal{A}$ in the case that $\mathcal{A}$ has a unary term operation which is an endomorphism of $\mathcal{A}$ at the same time is presented. A bijective mapping occurs in this condition and mono-unary algebras are used substantially.

Keywords: Algebra, direct limit, retract, operation, endomorphism.

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1. INTRODUCTION

In this article, we deal with universal algebras and the focus is on the notion of retract and direct limit construction. We demonstrate how a result for mono-unary algebras is also useful for algebras of other types, e.g. groups.

The importance of the notion of retract is well known and commonly appreciated in mathematics. This notion connects homomorphisms and subalgebras in some sense in universal algebra. There are many papers dealing with retracts of algebraic structures, see e.g. [8, 9]. The construction of the direct limit is a well-known method of building new algebras from given ones, see e.g. [2].

A universal algebra or, briefly, algebra $\mathcal{A}$, is a pair (A, F), where A is a non-void set and F is a set of finitary operations on A. The set F is not necessarily finite, and it can be void.

We denote by $\underrightarrow{\mathbf{L}}\mathcal{A}$ the class of all isomorphic copies of direct limits which can be obtained from the algebra $\mathcal{A}$ and we denote by $\mathbf{R}\mathcal{A}$ the set of all retracts of the algebra $\mathcal{A}$. Obviously, it is $\mathbf{R}\mathcal{A} \subseteq \underrightarrow{\mathbf{L}}\mathcal{A}$. We will say that $\mathcal{A}$ is an algebra with easy direct limits if every algebra from $\underrightarrow{\mathbf{L}}\mathcal{A}$ is isomorphic to a retract of $\mathcal{A}$. If $\mathcal{A}$ is finite, then $\mathcal{A}$ is an algebra with easy direct limits, cf. [6]. A class of infinite algebras with easy direct limits can be found in [5].

We will prove that if $\mathcal{A}$ is an algebra that has a term operation which is an endomorphism of $\mathcal{A}$ at the same time, then a special "diamond" mono-unary algebra can be constructed by direct limits, see Theorem 3.1. This "diamond" algebra can help to recognize, that $\mathcal{A}$ is not with easy direct limits. We will illustrate it on additive groups of integers and multiplicative group of rational numbers.

Mono-unary algebras are the most simple types of algebraic structures. They can be represented by oriented graphs with one outgoing arrow from every vertex. Basic terminology and some results can be found in monographs [1, 10, 13]. Remark that if the range of a function

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is a subset of its domain, then this function defines a mono-unary algebra. Mono-unary algebras with easy direct limits were studied in [5]. Among other things, it is proven there that a mono-unary algebra with easy direct limits

- is countable,
- has the number of retracts never equal to $\aleph_0$.

We study what does it mean that a term operation is an endomorphism in particular algebraic structures in the last section of this paper. Specifically, we deal with abelian groups, rings of characteristic zero with a unit, mono-unary and unary algebras.

2. PRELIMINARIES

Let $\mathbb{Z}$ be the set of all integers, $\mathbb{N}$ be the set of all positive integers and $\mathbb{N}_0$ be the set of all non-negative integers. Let A, B, C be non-empty sets and g, h be mappings, $g : A \to B, h : B \to C$. We denote by $g \circ h$ the mapping from A to C such that $(g \circ h)(a) = h(g(a))$ for each $a \in A$. Further, if $g : A \to A$, then g^0 denotes the identity mapping on A and if $k \in \mathbb{N}$, then $g^k = g^{k-1} \circ g$ by induction.

The notion of direct limit we apply by [2, §21].Let $\langle P, \leq \rangle$ be a directed partially ordered set, i.e., partially ordered set in which every finite subset has an upper bound. For each $p \in P$, let $\mathcal{A}_p = (A_p, F)$ be an algebra of some fixed type. Assume that if $p, q \in P,\ p \neq q$, then $A_p \cap A_q = \emptyset$. Suppose that for each pair of elements p and q in P with $p < q$, we have a homomorphism φ_{pq} of $\mathcal{A}_p$ into $\mathcal{A}_q$ such that $p < q < s$ implies that $\varphi_{ps} = \varphi_{pq} \circ \varphi_{qs}$. For each $p \in P$, suppose that φ_{pp} is the identity on A_p. The family $\{P, \mathcal{A}_p, \varphi_{pq}\}$ is said to be direct.

Assume that $p, q \in P$ and $x \in A_p,\ y \in A_q$. Put $x \equiv y$ if there exists $s \in P$ with $p \leq s,\ q \leq s$ such that $\varphi_{ps}(x) = \varphi_{qs}(y)$. Obviously, the relation $\equiv$ is an equivalence relation. For each $z \in \bigcup_{p \in P} A_p$ put $\overline{z} = \{t \in \bigcup_{p \in P} A_p : z \equiv t\}$. Denote $\overline{A} = \{\overline{z} : z \in \bigcup_{p \in P} A_p\}$.

Let $f \in F$ be an n-ary operation. Let $x_j \in A_{p_j}$, $1 \leq j \leq n$ and let s be an upper bound of p_j. Define $f(\overline{x_1}, ..., \overline{x_n}) = \overline{f(\varphi_{p_1 s}(x_1), ..., \varphi_{p_n s}(x_n))}$. Then the algebra $\overline{\mathcal{A}} = (\overline{A}, F)$ is said to be the direct limit of the direct family $\{P, \mathcal{A}_p, \varphi_{pq}\}$. We express this situation as follows

$$\{P, \mathcal{A}_p, \varphi_{pq}\} \longrightarrow \overline{\mathcal{A}}. \tag{2.1}$$

Note that in the category theory this construction corresponds to (directed) colimit. Let $\mathcal{A} = (A, F)$ and $\mathcal{B} = (B, F)$ be algebras. If $\mathcal{A}$ is isomorphic to $\mathcal{B}$, then we write $\mathcal{A} \cong \mathcal{B}$. Suppose that $\mathcal{B}$ is a subalgebra of $\mathcal{A}$. Then $\mathcal{B}$ is said to be a retract of $\mathcal{A}$ if there exists an endomorphism φ of $\mathcal{A}$ such that $\varphi(A) = B$ and $\varphi(b) = b$ for every $b \in B$. The mapping φ is called the retraction of $\mathcal{A}$.

If $\{P, \mathcal{A}_p, \varphi_{pq}\}$ is a family such that $\mathcal{A}_p \cong \mathcal{A}$, then this family is called the $\mathcal{A}$-uniform family. Every retract of $\mathcal{A}$ can be (up to isomorphism) obtained as a limit of an $\mathcal{A}$-uniform direct family, cf. e.g. [4]. If the direct limit of every $\mathcal{A}$-uniform direct family is isomorphic to a retract of $\mathcal{A}$, then we say that $\mathcal{A}$ is the algebra with easy direct limits.

We denote by $\underrightarrow{\mathbf{L}}\mathcal{A}$ the class of all isomorphic copies of direct limits of $\mathcal{A}$-uniform direct families. The set of term operations, in short terms, of the algebra $\mathcal{A}$ is denoted by $T(F)$ and it is the smallest set that

1. it contains each $f \in F$,
2. it contains all coordinate projections $p_i^n(a_1, ..., a_n) = a_i$, where $i, n \in \mathbb{N}, i \leq n$,
3. is closed under composition, i.e. if $h \in T(F)$ is m-ary operation, $m \in \mathbb{N}$, $f_1, ..., f_m \in T(F)$ are n-ary, $n \in \mathbb{N}_0$, then the operation $g : A^n \to A$ defined by
$$g(a_1, ...a_n) = h(f_1(a_1, ..., a_n), ..., f_m(a_1, ..., a_n)) \text{ for each } a_1, ..., a_n \in A$$
belongs to $T(F)$.

We finish this section with a note that the parentheses for F in (A, F) will be omitted for several specific types of algebras, e.g. groups and rings.

3. APPLICATION OF MONO-UNARY ALGEBRAS

Lemma 3.1. *Let φ be a homomorphism from an algebra (A, F) into (A', F) and $g \in T(F)$. Then φ is a homomorphism from the algebra (A, g) into (A', g).*

Proof. Assume that g is n-ary operation, $n \in \mathbb{N}$. We need to check that g is compatible with φ on the set A, i.e.

$$\varphi(g(a_1, ..., a_n)) = g(\varphi(a_1), ..., \varphi(a_n))$$

for all $a_1, ..., a_n \in A$. If $g \in F$ or g is a projection, then it is obvious.

Suppose that $h \in T(F)$ is m-ary, $f_1, ..., f_m \in T(F)$ are n-ary, $h, f_1, ..., f_m$ are compatible with φ and $g(a_1, ..., a_n) = h(f_1(a_1, ..., a_n), ..., f_m(a_1, ..., a_n))$. We obtain

$$\begin{aligned}\varphi(g(a_1, ..., a_n)) &= \varphi(h(f_1(a_1, ..., a_n), ..., f_m(a_1, ..., a_n)))\\ &= h(\varphi(f_1(a_1, ..., a_n)), ..., \varphi(f_m(a_1, ..., a_n)))\\ &= h(f_1(\varphi(a_1), ..., \varphi(a_n)), ..., f_m(\varphi(a_1), ..., \varphi(a_n))\\ &= g(\varphi(a_1), ..., \varphi(a_n)).\end{aligned}$$

□

Lemma 3.2. *Let $\mathcal{D} = (D, F)$ be an algebra and $g \in T(F)$. Suppose that $\{P, \mathcal{A}_p, \varphi_{pq}\}$ is a direct family and $\{P, \mathcal{A}_p, \varphi_{pq}\} \longrightarrow \overline{\mathcal{A}} = (\overline{A}, F)$.*

If $\overline{\mathcal{A}}$ is isomorphic to a retract of $\mathcal{D}$, then $(\overline{A}, g)$ is isomorphic to a retract of (D, g).

Proof. Assume that $\overline{\mathcal{A}}$ is isomorphic to a retract of $\mathcal{D}$. Then there exist

- $B \subseteq D$ and an endomorphism φ of $\mathcal{D}$ such that $\varphi(D) = B$ and $\varphi(b) = b$ for every $b \in B$,
- an isomorphism ψ from (B, F) onto $\overline{\mathcal{A}}$.

The mapping ψ is bijection and in view of the previous lemma ψ is the homomorphism from (B, g) into $(\overline{A}, g)$. So, ψ is the isomorphism from (B, g) onto $(\overline{A}, g)$. Moreover, φ is an endomorphism of (D, g). Thus φ is a retraction of (D, g). □

The proof of the following lemma follows from the definition.

Lemma 3.3. *Let $\mathcal{A} = (A, F)$ and f be an endomorphism of the algebra $\mathcal{A}$. Suppose that*

1. *$A_i = \{(a, i) | a \in A\}$ for each $i \in \mathbb{N}$,*
2. *$g((a_1, i), ..., (a_n, i)) = (g(a_1, ..., a_n), i)$ for each n-ary operation $g \in F, i, n \in \mathbb{N}, a_1, ..., a_n \in A$,*
3. *$\mathcal{A}_i = (A_i, F)$ for each $i \in \mathbb{N}$,*
4. *$\varphi_{i,j}(a, i) = (f^{j-i}(a), j)$ for all $i \leq j, i, j \in \mathbb{N}$.*

Then $\{\mathbb{N}, \mathcal{A}_i, \varphi_{i,j}\}$ is $\mathcal{A}$-uniform direct system. If $\{\mathbb{N}, \mathcal{A}_i, \varphi_{i,j}\} \longrightarrow \overline{\mathcal{A}}$ and $\mathcal{A}$ is with easy direct limits, then $\overline{\mathcal{A}}$ is isomorphic to a retract of $\mathcal{A}$.

The algebra $\overline{\mathcal{A}}$ from the previous lemma is fully described in the case of mono-unary algebras for f equal to the fundamental operation of this algebra, cf.[5]. We repeat this description after the short introduction to mono-unary algebras.

Let $A \neq \emptyset$ and h be a unary mapping from A into A. The couple $(A, \{h\})$ is called mono-unary algebra; in simplified way, we write (A, h). Such algebra can be visualised as a directed graph; the vertices are elements of A and for all $a \in A$ there is a directed edge from a to its image $h(a)$. Note that there is exactly one out-edge at each vertex. The terms of cycle, length

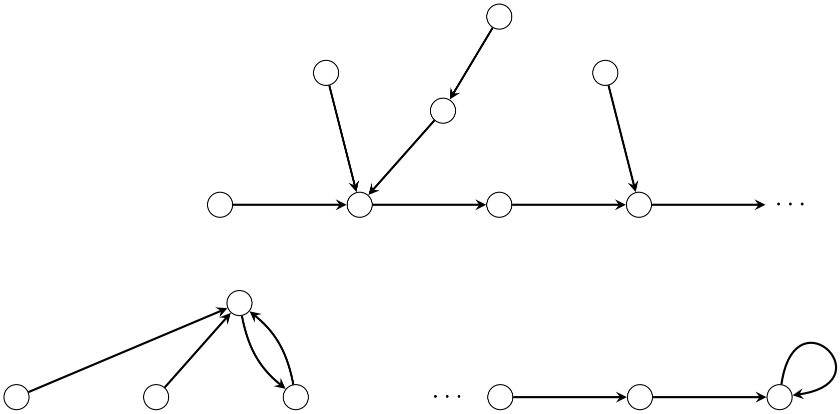

FIGURE 1. A mono-unary algebra

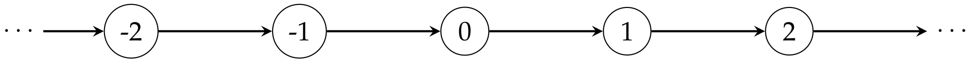

FIGURE 2. The algebra Z

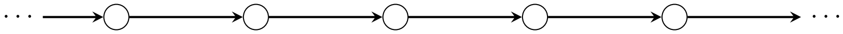

FIGURE 3. A line

of a cycle, cyclic element, connected monounary algebra are intuitively clear from a graph visualisation. Formal definitions can be found in [1, 9, 13].

Example 3.1. *A mono-unary algebra which is not connected is depicted in Figure 1. It consists of 3 (connected) components. Two components have a cycle, one is without a cycle. The component with 1-element cycle is infinite and this component is a mono-unary algebra with infinitely many different subalgebras.*

We denote by Z the mono-unary algebra that is defined on the set of all integer numbers with the successor function, see Figure 2.

A mono-unary algebra (A, h) is called a line if it is isomorphic to the algebra Z, see Figure 3.

The next assertion is obvious.

Lemma 3.4. *Let (A, h) be a mono-unary algebra. Then the following statements are equivalent:*

(1) (A, h) is a line,
(2) (A, h) is connected, A is infinite and the operation h is bijective,
(3) (A, h) is connected without a cycle and the operation h is bijective.

Lemma 3.5. *Let (A, h) be a connected mono-unary algebra. If (A, h) contains a subalgebra C such that C is a cycle or C is a line, then C is a retract of (A, h).*

Proof. Put $\varphi(c) = c$ for $c \in C$. Let $a \in A \setminus C$. Then there is $n \in \mathbb{N}$ such that $h^n(a) \in C$ and $h^{n-1}(a) \notin C$. The set C contains exactly one element b such that $h^n(b) = h^n(a)$. Put $\varphi(a) = b$. We have that $\varphi(A) = C$ and φ is the retraction. □

Now we come to the essential assignment in this section. Let I be a nonempty set. For each $i \in I$ let (B_i, h) be a mono-unary algebra. We denote by $\sum_{i \in I}(B_i, h)$ a mono-unary algebra

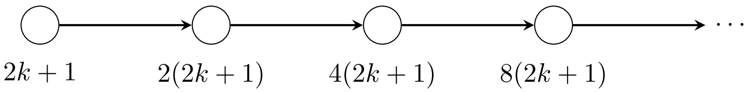

FIGURE 4. Infinite components of $(\mathbb{Z}, f)$, $k \in \mathbb{Z}$

which is a disjoint union of algebras (B_i, h), $i \in I$. Let $(A, h) = \sum_{i \in I}(B_i, h)$ and (B_i, h) be connected for all $i \in I$. Let $i \in I$. If (B_i, h) contains a cycle of length k for some $k \in \mathbb{N}$, then we denote by (C_i, h) a cycle of length k. Else we denote by (C_i, h) a line. Put

$$(A, h)^\diamond = \sum_{i \in I}(C_i, h).$$

If $\mathcal{A} = (A, h)$ and $f = h$ in Lemma 3.3, then $\overline{(A, h)} \cong (A, h)^\diamond$ according to [3], Lemma 4.

Theorem 3.1. *Let $\mathcal{A} = (A, F)$ and $f \in T(F)$ be unary term such that f is an endomorphism of the algebra $\mathcal{A}$. Suppose that $\{\mathbb{N}, \mathcal{A}_i, \varphi_{i,j}\}$ is $\mathcal{A}$-uniform direct system from Lemma 3.3 and $\{\mathbb{N}, \mathcal{A}_i, \varphi_{i,j}\} \longrightarrow \overline{\mathcal{A}} = (\overline{A}, F)$. Then $(\overline{A}, f) \cong (A, f)^\diamond$.*

If $\mathcal{A}$ is an algebra with easy direct limits, then the mono-unary algebra $(A, f)^\diamond$ is isomorphic to a retract of (A, f).

Proof. Obviously $\{\mathbb{N}, (A_i, f), \varphi_{i,j}\}$ is (A, f)-uniform direct system of mono-unary algebras and $\{\mathbb{N}, (A_i, f), \varphi_{i,j}\} \longrightarrow (\overline{A}, f)$. In view of Lemma 4 of [3] is $(\overline{A}, f) \cong (A, f)^\diamond$.

Suppose that $\mathcal{A} = (A, F)$ is an algebra with easy direct limits. Then $\overline{\mathcal{A}}$ is isomorphic to a retract of $\mathcal{A}$. It yields that $(\overline{A}, f)$ is isomorphic to a retract of (A, f) according to Lemma 3.2. ☐

Corollary 3.1. *Let $\mathcal{A} = (A, F)$ and $f \in T(F)$ be unary term such that f is an endomorphism of the algebra $\mathcal{A}$. Then*

1. *there exists $\mathcal{B} = (B, F) \in \underrightarrow{\mathbf{L}}\mathcal{A}$ such that f is bijective on B,*
2. *if $\mathcal{A}$ is an algebra with easy direct limits, then there exists $\mathcal{B} = (B, F) \in \mathbf{R}\mathcal{A}$ such that f is bijective on B.*

Example 3.2. *The operation $f(x) = x + x$ is a term of $\{+\}$, where $+$ is usual binary addition operation. The additive group of integers $(\mathbb{Z}, +, -, 0)$ is commutative and therefore f is an endomorphism of this group. Let us look at the mono-unary algebra $(\mathbb{Z}, f)$. It consists of infinitely many components. One of them is $\{0\}$, others are infinite and isomorphic to each other, see Figure 4. Every infinite component is generated by an odd number, therefore no subalgebra of $(\mathbb{Z}, f)$ contains a line.*

The algebra $(\mathbb{Z}, f)^\diamond$ consists of one 1-element cycle and infinitely many lines according to the definition. Thus $(\mathbb{Z}, f)^\diamond$ is not isomorphic to a retract of $(\mathbb{Z}, f)$. Theorem 3.1 implies that the group $(\mathbb{Z}, +, -, 0)$ is not an algebra with easy direct limits.

Example 3.3. *The operation $g(x) = x \cdot x$ is a term of $\{\cdot\}$, where $\cdot$ is usual binary multiplication operation.*

The multiplicative monoid of real numbers $(\mathbb{R}, \cdot, 1)$ is commutative and therefore g is an endomorphism of this group at the same time. The mono-unary algebra $(\mathbb{R}, g)$ consists of uncountable many components. Finite ones are $\{0\}$ and $\{-1, 1\}$, others are mutually isomorphic, see Figure 5. Every infinite component contains one line as its subalgebra. The algebra $(\mathbb{R}, g)^\diamond$ consists of two 1-element cycles and lines. (Note that it is isomorphic to $(\mathbb{R}_0^+, g)$, where $\mathbb{R}_0^+$ is the set of all non-negative real numbers.) Thus $(\mathbb{R}, g)^\diamond$ is isomorphic to a retract of $(\mathbb{R}, g)$ according to Lemma 3.5. Therefore Theorem 3.1 does not give the answer whether $(\mathbb{R}, \cdot, 1)$ is the algebra with easy direct limits.

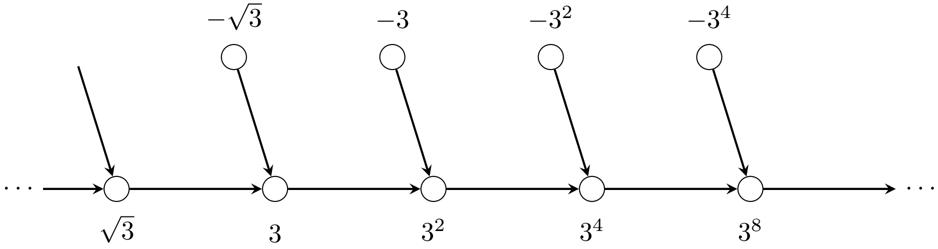

FIGURE 5. The component of $(\mathbb{R}, g)$ which contains the number 3

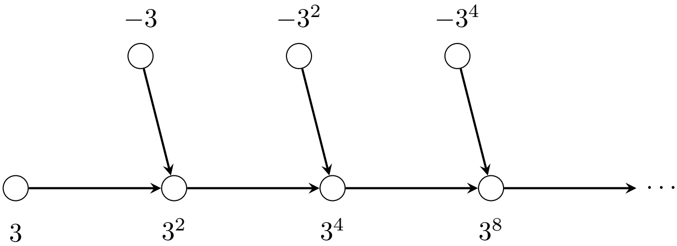

FIGURE 6. The component of $(\mathbb{Q}, g)$ which contains the number 3

Consider the monoid $(\mathbb{Q}, \cdot, 1)$. *The mono-unary algebra* $(\mathbb{Q}, g)$ *consists of infinitely many components too. Infinite ones are isomorphic to each other, see Figure 6. There is no line as a subalgebra. The algebra* $(\mathbb{Q}, g)^\diamond$ *is not isomorphic to a retract of* $(\mathbb{Q}, g)$. *(In fact, the* $(\mathbb{Q} \setminus \{0\}, g)^\diamond$ *is isomorphic to the algebra* $(\mathbb{Z}, f)^\diamond$ *from the previous example.) Therefore* $(\mathbb{Q}, \cdot, 1)$ *is not the algebra with easy direct limits according to Theorem 3.1.*

4. UNARY TERMS WHICH ARE ENDOMORPHISMS

Let $\mathcal{A} = (A, F)$ be an algebra. The identity mapping is a term operation, since it is a projection. It is an endomorphism of $\mathcal{A}$ at the same time. If $\varphi, \psi \in T(F)$ are endomorphisms of $\mathcal{A}$, then obviously $\varphi \circ \psi \in T(F)$ and $\varphi \circ \psi$ is an endomorphism of $\mathcal{A}$. We will see that

- all unary term operations are endomorphisms of $\mathcal{A}$ in the case of abelian groups and mono-unary algebras;
- the identity mapping is the only term operation that is an endomorphism of $\mathcal{A}$ in the case of rings of characteristic zero with 1;
- unary term operations which are endomorphisms of $\mathcal{A}$ are closely linked to the structure of $\mathcal{A}$ in the case of unary algebras.

Proposition 4.1. *Let* $(G, +, -, 0)$ *be an abelian group. Then every unary term is an endomorphism of* $(G, +, -, 0)$.

Proof. Every unary term of an additive group has a form $f(x) = kx$, for some $k \in \mathbb{Z}$. Therefore $f(0) = 0$ for $0 \in G$, Assume that $g_1, g_2 \in G$. Then

$$f(-g_1) = k(-g_1) = -(kg_1) = -f(g_1)$$

according to group properties. Further,

$$f(g_1 + g_2) = k(g_1 + g_2) = kg_1 + kg_2$$

since $(G, +, -, 0)$ is abelian. ☐

Proposition 4.2. *Let* $(R, +, -, \cdot, 0, 1)$ *be a ring of characteristic zero with* 1*. Then a unary term is an endomorphism of* $(R, +, -, , \cdot, 0, 1)$ *if and only if it is the identity operation.*

Proof. It is obvious that every unary term of a ring has a form

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0,$$

where $a_i \in \mathbb{Z}, i \in \{1, \ldots, n\}$. Every ring with characteristic zero contains a subalgebra which is isomorphic to the ring of integers $(\mathbb{Z}, +, \cdot, -, 0, 1)$. Without loss of generality suppose that $\mathbb{Z} \subseteq R$.

Assume that f is an endomorphism of $(R, +, \cdot, -, 0, 1)$. Hence f is an endomorphism of the group $(R, +, -, 0)$ and $f(1) = 1$. Thus $f(m) = m$ is valid for every $m \in \mathbb{Z}$. Therefore $a_0 = 0$.

We obtained

$$\begin{aligned} a_0 = f(0) &= 0, \\ a_n + a_{n-1} + \ldots + a_1 = f(1) &= 1, \\ a_n 2^n + a_{n-1}2^{n-1} + \ldots + a_1 2 = f(2) &= 2, \\ &\vdots \\ a_n n^n + a^{n-1}n^{n-1} + \ldots + a_1 n = f(n) &= n. \end{aligned}$$

Dividing $k-$th equation by k for each $k \in \{1, \ldots, n\}$, we obtain a system of linear equations:

$$\text{(4.2)} \qquad \left\{ \begin{array}{ccccccccc} a_n & + & a_{n-1} & + & \ldots & + & a_2 & + & a_1 = 1 \\ a_n 2^{n-1} & + & a_{n-1}2^{n-2} & + & \ldots & + & a_2 2 & + & a_1 = 1 \\ \vdots & & & & & & & & \\ a_n n^{n-1} & + & a_{n-1}n^{n-2} & + & \ldots & + & a_2 n & + & a_1 = 1 \end{array} \right. .$$

The coefficient matrix has the form:

$$\begin{bmatrix} 1 & 1 & \ldots & 1 & 1 \\ 2^{n-1} & 2^{n-2} & \ldots & 2 & 1 \\ \vdots & & & & \\ n^{n-1} & n^{n-2} & \ldots & n & 1 \end{bmatrix}.$$

This is a Vandermond matrix with non-zero determinant, which implies that the system of linear equations (4.2) has exactly one solution. Hence,

$$a_1 = 1, a_2 = \ldots = a_n = 0$$

is the only solution. This completes the proof. ☐

Proposition 4.3. *Let* (A, h) *be a mono-unary algebra. Then every unary term is an endomorphism of* (A, h)*.*

Proof. Let g be a unary term over F. Then $g = h^k$ for some $k \in \mathbb{N}_0$. Assume that $a \in A$. We have

$$g(h(a)) = h^k(h(a)) = h^{k+1}(a) = h(h^k(a)) = h(g(a)).$$

☐

Next example demonstrates that there are algebras which have a unary term that is not an endomorphism and at the same time they have at least two unary terms that are endomorphisms.

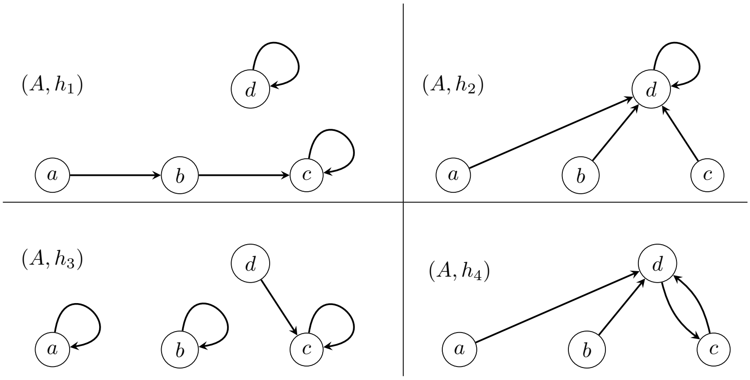

FIGURE 7. Mono-unary algebras from Example 4.4

Example 4.4. *Let $A = \{a, b, c, d\}$ and unary operations $h_1 - h_4$ on A are given by Table 1.*

	h_1	h_2	h_3	h_4
a	b	d	a	d
b	c	d	b	d
c	c	d	c	d
d	d	d	c	c

TABLE 1. Operations $h_1 - h_4$

Consider unary algebra $\mathcal{A} = (A, \{h_1, h_2, h_3, h_4\})$. We name three unary terms different from the identity that are endomorphisms of $\mathcal{A}$ and three unary terms that are not endomorphisms of $\mathcal{A}$. The term h_1 is an endomorphism of $\mathcal{A}$, since it is an endomorphism of mono-unary algebras (A, h_i) for $i = 1, 2, 3, 4$; algebras (A, h_i) are in Figure 7. The term h_1^2 is an endomorphism of $\mathcal{A}$ since the composition of endomorphisms is an endomorphism. Note that for $k \in \mathbb{N}, k > 2$ is $h_1^k = h_1^2$ on A. Further, it is easy to check that h_4^2 is an endomorphism of (A, h_i) for $i = 1, 2, 3, 4$ and therefore this term is an endomorphism of $\mathcal{A}$. Terms h_2, h_3, h_4 are not endomorphisms of $\mathcal{A}$.

The following assertion characterizes all unary algebras which have the property that all unary terms are endomorphisms at the same time. It follows from definitions.

Proposition 4.4. *Let $\mathcal{A} = (A, F)$ be a unary algebra. Then $\varphi : A \to A$ is an endomorphism of $\mathcal{A}$ if and only if it is an endomorphism of the mono-unary algebra (A, h) for each $h \in F$. Further, the following properties are equivalent:*

1. *every unary term of $\mathcal{A}$ is an endomorphism of $\mathcal{A}$,*
2. *if $h \in F$, then h is an endomorphism of $\mathcal{A}$.*

The last statement helps recognise unary algebras which have a constant term operation that is an endomorphism at the same time.

Proposition 4.5. *Let (A, F) be a unary algebra and $F = \{h_i, i \in I\}$, where h_i is a unary operation for each $i \in I$.*
Suppose that $g \in T(F)$ is unary and $a^\star \in A$ are such that

1. *$h_i(a^\star) = a^\star$ for each $i \in I$,*
2. *there exists $k \in \mathbb{N}$ such that $g^k(a) = a^\star$ for each $a \in A$.*

Then $g^k \in T(F)$ *and* g^k *is an endomorphism of* (A, F).

Proof. The condition (2) says that the mapping g^k is constant on A. We have

$$g^k(h_i(a)) = a^\star = h_i(a^\star) = h_i(g^k(a)).$$

□

Example 4.5. *Consider the unary algebra* $(A, \{h_1, h_3, h_4^2\})$*, where* A, h_1, h_3, h_4 *are from the previous example. We denote by* g *the constant operation equal to* c*. Then* $g = h_3 \circ h_4^2 \in T(\{h_1, h_3, h_4^2\})$*. If we put* $a^\star = c$ *and* $k = 1$*, then suppositions of Proposition 4.5 are valid. Thus* g *is an endomorphism of* $(A, \{h_1, h_3, h_4^2\})$.

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