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Research Article

3rd International Conference: Constructive Mathematical Analysis (ICCMA'25)

TUNCER ACAR* AND METIN TURGAY

ABSTRACT. The guest editors provides an overview of the 3rd International Conference: Constructive Mathematical Analysis (ICCMA 2025), held on 2–5 July 2025 at Selçuk University, Konya, Türkiye, summarizing its objectives, scope, scientific aims, and the key highlights of the event. The conference brought together researchers and experts in approximation theory, functional analysis, operator theory, and related fields to exchange ideas, present new results, and foster collaboration within the constructive mathematics community. The note briefly introduces the papers included in this special issue of the *ALTAY Conference Proceedings in Mathematics*, which reflect current advances in constructive mathematics, approximation theory, nonlinear analysis, and their diverse applications.

1. REPORT ON THE CONFERENCE

The *3rd International Conference: Constructive Mathematical Analysis (ICCMA 2025)* was held on 2–5 July 2025 at Selçuk University, Konya, Türkiye. The event provided a global forum for researchers, academicians, and young scientists to exchange ideas, present recent findings, and discuss current challenges in constructive mathematical analysis and its applications. Altogether, the conference hosted approximately 180 participants representing 31 countries and included 170 oral and 8 poster presentations.

The scientific program covered a wide range of topics, including approximation theory, functional analysis, sampling-type operators, fixed-point theory, Fourier analysis, fractal calculus, and mathematical modelling. The plenary lectures were delivered by distinguished speakers who provided deep insights into current trends in analysis and its interdisciplinary applications. The plenary speakers were:

i. Prof. Francesco Altomare from University of Bari, Italy
ii. Prof. Erdal Karapnar from Atlm University, Türkiye
iii. Prof. Harun Karslı from Bolu Abant Izzet Baysal University, Türkiye
iv. Prof. Mohammad Sal Moslehian from Ferdowsi University of Mashhad, Iran
v. Prof. Ioan Raa from Technical University of Cluj-Napoca, Romania
vi. Prof. Gianluca Vinti from University of Perugia, Italy
vii. Prof. Xiaoming Wang from Eastern Institute of Technology and Missouri University of Science and Technology, China and USA
viii. Prof. Ferenc Weisz from Eötvös University, Hungary

Each lecture addressed contemporary problems in mathematics, emphasizing both theoretical advances and practical applications.

The conference was organized in a presence format and featured special sessions on

S1: Positive Approximation Processes and Applications

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S2: Approximation by Sampling type Operators and Applications
S3: Nonlinear Analysis, Fixed Point Theory and Applications
S4: Fourier Analysis and Applications
S5: Fractal Calculus and its Applications
S6: All other topics in mathematics and statistics.

The event was chaired by Prof. Tuncer Acar, with the support of the Organizing and Scientific Committees. Further details, including the complete program and abstracts, are available on the official website: `https://iccma.selcuk.edu.tr`.

2. Introducing the Special Issue

The papers published in this special issue of the *ALTAY Conference Proceedings in Mathematics* originate from the *3rd International Conference: Constructive Mathematical Analysis (ICCMA 2025)*, held at Selçuk University, Konya, Türkiye. The proceedings aim to ensure the scientific continuity between the conference presentations and the peer-reviewed literature, providing an official record of selected contributions reflecting the main themes of the event.

All submissions to this issue were invited from authors who presented their work at ICCMA 2025. Each manuscript underwent a rigorous peer-review process coordinated by the Guest Editors and the Scientific Committee. Every paper was evaluated by at least two independent referees for originality, correctness, and alignment with the conference scope. Only those meeting the journals editorial standards and the ethical publication guidelines were accepted after revision.

The following papers were accepted for publication in this issue:

i. Emília Halušková and Małgorzata Jastrzębska, *Mono-unary condition for algebras with easy direct limits*. This paper establishes a structural condition for an algebra $\mathcal{A}$ where every algebra isomorphic to its retract can be obtained as a direct limit, showing that this holds when $\mathcal{A}$ admits a unary term operation acting as an endomorphism, and highlighting the essential role of bijective mappings and mono-unary algebras in this framework.

ii. Ferenc Weisz, *Lebesgue points and summability of higher dimensional Fourier transforms*. This paper generalizes Lebesgues theorem to higher-dimensional settings and diverse summability methods within Wiener amalgam spaces.

iii. Harun Karsli, *Wavelet-based approximation operators: applications to bivariate functions and digital image processing*. This paper construct and investigate bivariate case of these operators wavelet based approximation operators and gives applications in image processing.

iv. Pathaithep Kumrod and Wutiphol Sintunavarat, *On new ϕ-fixed point results involving discontinuous control functions with the effectively example and its applications*. This paper develops ϕ-fixed point results with discontinuous control functions, offering generalized frameworks and applications to integral equations.

v. Barkat Ali Bhayo, *Distrotion and quasisymmetric functions in quasiconformal mappings*. This paper studies the applications of special functions and quasisymmetry in quasiconformal mappings and estimates the distances between the image points of quasiconformal mappings under variousmetrics.

vi. Cristina Maria Păcurar and Mirela Adriana Târnoveanu, *Mappings contracting perimeters of triangles in perturbed metric spaces*. This paper introduces perturbed mappings contracting perimeters of triangles and proves new fixed point theorems extending classical results.

vii. Muhammad Abubakar Isah and Ahmad Muhammad, *Propagation of solitons and nonlinear behavior in nonlinear power law fibers*. This paper analyzes soliton propagation in nonlinear optical fibers governed by the complex GinzburgLandau equation with power-law nonlinearity.

viii. Michele Piconi and Gianluca Vinti, *Semi-discrete sampling operators acting on function spaces*. This paper provides an overview of semi-discrete Durrmeyer-type sampling operators, including convergence, rates of approximation, and modular results in Orlicz spaces.

All these papers have been reviewed according to the standards of the ALTAY Conference Proceedings in Mathematics, ensuring a high level of scientific rigor and a coherent representation of the topics covered at *ICCMA 2025*.

3. Acknowledgements

The Guest Editors would like to express their sincere gratitude to the members of the Organizing and Scientific Committees of *ICCMA 2025* for their dedicated efforts in ensuring the success of the conference and the quality of its scientific program. Special thanks are extended to all referees and reviewers for their careful evaluations, constructive comments, and valuable time, which greatly improved the quality of the papers published in this volume. The editors also acknowledge the support and professional assistance of the ALTAY editorial staff throughout the publication process.

The conference and this special issue were financially and institutionally supported by the Scientific Research Projects Coordinatorship of Selçuk University, The Scientific and Technological Research Council of Türkiye (TÜBTAK), the Republic of Turkey Ministry of Youth and Sports, and BEYSU. Their contributions are gratefully acknowledged.

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Research Article

Mono-unary condition for algebras with easy direct limits

EMÍLIA HALUŠKOVÁ* AND MAŁGORZATA JASTRZĘBSKA

ABSTRACT. Let $\mathcal{A}$ be an algebra such that exactly algebras isomorphic to a retract of $\mathcal{A}$ can be constructed from $\mathcal{A}$ by direct limits. One condition which is satisfied for $\mathcal{A}$ in the case that $\mathcal{A}$ has a unary term operation which is an endomorphism of $\mathcal{A}$ at the same time is presented. A bijective mapping occurs in this condition and mono-unary algebras are used substantially.

Keywords: Algebra, direct limit, retract, operation, endomorphism.

2020 Mathematics Subject Classification: 08B25, 08A35, 08A40, 08A60, 33E99, 40J99.

1. INTRODUCTION

In this article, we deal with universal algebras and the focus is on the notion of retract and direct limit construction. We demonstrate how a result for mono-unary algebras is also useful for algebras of other types, e.g. groups.

The importance of the notion of retract is well known and commonly appreciated in mathematics. This notion connects homomorphisms and subalgebras in some sense in universal algebra. There are many papers dealing with retracts of algebraic structures, see e.g. [8, 9]. The construction of the direct limit is a well-known method of building new algebras from given ones, see e.g. [2].

A universal algebra or, briefly, algebra $\mathcal{A}$, is a pair (A, F), where A is a non-void set and F is a set of finitary operations on A. The set F is not necessarily finite, and it can be void.

We denote by $\underrightarrow{\mathbf{L}}\mathcal{A}$ the class of all isomorphic copies of direct limits which can be obtained from the algebra $\mathcal{A}$ and we denote by $\mathbf{R}\mathcal{A}$ the set of all retracts of the algebra $\mathcal{A}$. Obviously, it is $\mathbf{R}\mathcal{A} \subseteq \underrightarrow{\mathbf{L}}\mathcal{A}$. We will say that $\mathcal{A}$ is an algebra with easy direct limits if every algebra from $\underrightarrow{\mathbf{L}}\mathcal{A}$ is isomorphic to a retract of $\mathcal{A}$. If $\mathcal{A}$ is finite, then $\mathcal{A}$ is an algebra with easy direct limits, cf. [6]. A class of infinite algebras with easy direct limits can be found in [5].

We will prove that if $\mathcal{A}$ is an algebra that has a term operation which is an endomorphism of $\mathcal{A}$ at the same time, then a special "diamond" mono-unary algebra can be constructed by direct limits, see Theorem 3.1. This "diamond" algebra can help to recognize, that $\mathcal{A}$ is not with easy direct limits. We will illustrate it on additive groups of integers and multiplicative group of rational numbers.

Mono-unary algebras are the most simple types of algebraic structures. They can be represented by oriented graphs with one outgoing arrow from every vertex. Basic terminology and some results can be found in monographs [1, 10, 13]. Remark that if the range of a function

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is a subset of its domain, then this function defines a mono-unary algebra. Mono-unary algebras with easy direct limits were studied in [5]. Among other things, it is proven there that a mono-unary algebra with easy direct limits

- is countable,
- has the number of retracts never equal to $\aleph_0$.

We study what does it mean that a term operation is an endomorphism in particular algebraic structures in the last section of this paper. Specifically, we deal with abelian groups, rings of characteristic zero with a unit, mono-unary and unary algebras.

2. Preliminaries

Let $\mathbb{Z}$ be the set of all integers, $\mathbb{N}$ be the set of all positive integers and $\mathbb{N}_0$ be the set of all non-negative integers. Let A, B, C be non-empty sets and g, h be mappings, $g : A \to B, h : B \to C$. We denote by $g \circ h$ the mapping from A to C such that $(g \circ h)(a) = h(g(a))$ for each $a \in A$. Further, if $g : A \to A$, then g^0 denotes the identity mapping on A and if $k \in \mathbb{N}$, then $g^k = g^{k-1} \circ g$ by induction.

The notion of direct limit we apply by [2, §21].Let $\langle P, \leq \rangle$ be a directed partially ordered set, i.e., partially ordered set in which every finite subset has an upper bound. For each $p \in P$, let $\mathcal{A}_p = (A_p, F)$ be an algebra of some fixed type. Assume that if $p, q \in P,\ p \neq q$, then $A_p \cap A_q = \emptyset$. Suppose that for each pair of elements p and q in P with $p < q$, we have a homomorphism φ_{pq} of $\mathcal{A}_p$ into $\mathcal{A}_q$ such that $p < q < s$ implies that $\varphi_{ps} = \varphi_{pq} \circ \varphi_{qs}$. For each $p \in P$, suppose that φ_{pp} is the identity on A_p. The family $\{P, \mathcal{A}_p, \varphi_{pq}\}$ is said to be direct.

Assume that $p, q \in P$ and $x \in A_p,\ y \in A_q$. Put $x \equiv y$ if there exists $s \in P$ with $p \leq s,\ q \leq s$ such that $\varphi_{ps}(x) = \varphi_{qs}(y)$. Obviously, the relation $\equiv$ is an equivalence relation. For each $z \in \bigcup_{p \in P} A_p$ put $\overline{z} = \{t \in \bigcup_{p \in P} A_p : z \equiv t\}$. Denote $\overline{A} = \{\overline{z} : z \in \bigcup_{p \in P} A_p\}$.

Let $f \in F$ be an n-ary operation. Let $x_j \in A_{p_j}$, $1 \leq j \leq n$ and let s be an upper bound of p_j. Define $f(\overline{x_1}, ..., \overline{x_n}) = \overline{f(\varphi_{p_1 s}(x_1), ..., \varphi_{p_n s}(x_n))}$. Then the algebra $\overline{\mathcal{A}} = (\overline{A}, F)$ is said to be the direct limit of the direct family $\{P, \mathcal{A}_p, \varphi_{pq}\}$. We express this situation as follows

$$\{P, \mathcal{A}_p, \varphi_{pq}\} \longrightarrow \overline{\mathcal{A}}. \tag{2.1}$$

Note that in the category theory this construction corresponds to (directed) colimit. Let $\mathcal{A} = (A, F)$ and $\mathcal{B} = (B, F)$ be algebras. If $\mathcal{A}$ is isomorphic to $\mathcal{B}$, then we write $\mathcal{A} \cong \mathcal{B}$. Suppose that $\mathcal{B}$ is a subalgebra of $\mathcal{A}$. Then $\mathcal{B}$ is said to be a retract of $\mathcal{A}$ if there exists an endomorphism φ of $\mathcal{A}$ such that $\varphi(A) = B$ and $\varphi(b) = b$ for every $b \in B$. The mapping φ is called the retraction of $\mathcal{A}$.

If $\{P, \mathcal{A}_p, \varphi_{pq}\}$ is a family such that $\mathcal{A}_p \cong \mathcal{A}$, then this family is called the $\mathcal{A}$-uniform family. Every retract of $\mathcal{A}$ can be (up to isomorphism) obtained as a limit of an $\mathcal{A}$-uniform direct family, cf. e.g. [4]. If the direct limit of every $\mathcal{A}$-uniform direct family is isomorphic to a retract of $\mathcal{A}$, then we say that $\mathcal{A}$ is the algebra with easy direct limits.

We denote by $\underrightarrow{\mathbf{L}}\mathcal{A}$ the class of all isomorphic copies of direct limits of $\mathcal{A}$-uniform direct families. The set of term operations, in short terms, of the algebra $\mathcal{A}$ is denoted by $T(F)$ and it is the smallest set that

(1) it contains each $f \in F$,
(2) it contains all coordinate projections $p_i^n(a_1, ..., a_n) = a_i$, where $i, n \in \mathbb{N}, i \leq n$,
(3) is closed under composition, i.e. if $h \in T(F)$ is m-ary operation, $m \in \mathbb{N}$, $f_1, ..., f_m \in T(F)$ are n-ary, $n \in \mathbb{N}_0$, then the operation $g : A^n \to A$ defined by
$$g(a_1, ...a_n) = h(f_1(a_1, ..., a_n), ..., f_m(a_1, ..., a_n)) \text{ for each } a_1, ..., a_n \in A$$
belongs to $T(F)$.

We finish this section with a note that the parentheses for F in (A, F) will be omitted for several specific types of algebras, e.g. groups and rings.

3. Application of mono-unary algebras

Lemma 3.1. *Let φ be a homomorphism from an algebra (A, F) into (A', F) and $g \in T(F)$. Then φ is a homomorphism from the algebra (A, g) into (A', g).*

Proof. Assume that g is n-ary operation, $n \in \mathbb{N}$. We need to check that g is compatible with φ on the set A, i.e.

$$\varphi(g(a_1, ..., a_n)) = g(\varphi(a_1), ..., \varphi(a_n))$$

for all $a_1, ..., a_n \in A$. If $g \in F$ or g is a projection, then it is obvious.

Suppose that $h \in T(F)$ is m-ary, $f_1, ..., f_m \in T(F)$ are n-ary, $h, f_1, ..., f_m$ are compatible with φ and $g(a_1, ..., a_n) = h(f_1(a_1, ..., a_n), ..., f_m(a_1, ..., a_n))$. We obtain

$$\begin{aligned}\varphi(g(a_1, ..., a_n)) &= \varphi(h(f_1(a_1, ..., a_n), ..., f_m(a_1, ..., a_n)))\\ &= h(\varphi(f_1(a_1, ..., a_n)), ..., \varphi(f_m(a_1, ..., a_n)))\\ &= h(f_1(\varphi(a_1), ..., \varphi(a_n)), ..., f_m(\varphi(a_1), ..., \varphi(a_n))\\ &= g(\varphi(a_1), ..., \varphi(a_n)).\end{aligned}$$

□

Lemma 3.2. *Let $\mathcal{D} = (D, F)$ be an algebra and $g \in T(F)$. Suppose that $\{P, \mathcal{A}_p, \varphi_{pq}\}$ is a direct family and $\{P, \mathcal{A}_p, \varphi_{pq}\} \longrightarrow \overline{\mathcal{A}} = (\overline{A}, F)$.*

If $\overline{\mathcal{A}}$ is isomorphic to a retract of $\mathcal{D}$, then $(\overline{A}, g)$ is isomorphic to a retract of (D, g).

Proof. Assume that $\overline{\mathcal{A}}$ is isomorphic to a retract of $\mathcal{D}$. Then there exist

- $B \subseteq D$ and an endomorphism φ of $\mathcal{D}$ such that $\varphi(D) = B$ and $\varphi(b) = b$ for every $b \in B$,
- an isomorphism ψ from (B, F) onto $\overline{\mathcal{A}}$.

The mapping ψ is bijection and in view of the previous lemma ψ is the homomorphism from (B, g) into $(\overline{A}, g)$. So, ψ is the isomorphism from (B, g) onto $(\overline{A}, g)$. Moreover, φ is an endomorphism of (D, g). Thus φ is a retraction of (D, g). □

The proof of the following lemma follows from the definition.

Lemma 3.3. *Let $\mathcal{A} = (A, F)$ and f be an endomorphism of the algebra $\mathcal{A}$. Suppose that*

(1) $A_i = \{(a, i) | a \in A\}$ for each $i \in \mathbb{N}$,
(2) $g((a_1, i), ..., (a_n, i)) = (g(a_1, ..., a_n), i)$ for each n-ary operation $g \in F, i, n \in \mathbb{N}, a_1, ..., a_n \in A$,
(3) $\mathcal{A}_i = (A_i, F)$ for each $i \in \mathbb{N}$,
(4) $\varphi_{i,j}(a, i) = (f^{j-i}(a), j)$ for all $i \leq j, i, j \in \mathbb{N}$.

Then $\{\mathbb{N}, \mathcal{A}_i, \varphi_{i,j}\}$ is $\mathcal{A}$-uniform direct system. If $\{\mathbb{N}, \mathcal{A}_i, \varphi_{i,j}\} \longrightarrow \overline{\mathcal{A}}$ and $\mathcal{A}$ is with easy direct limits, then $\overline{\mathcal{A}}$ is isomorphic to a retract of $\mathcal{A}$.

The algebra $\overline{\mathcal{A}}$ from the previous lemma is fully described in the case of mono-unary algebras for f equal to the fundamental operation of this algebra, cf.[5]. We repeat this description after the short introduction to mono-unary algebras.

Let $A \neq \emptyset$ and h be a unary mapping from A into A. The couple $(A, \{h\})$ is called mono-unary algebra; in simplified way, we write (A, h). Such algebra can be visualised as a directed graph; the vertices are elements of A and for all $a \in A$ there is a directed edge from a to its image $h(a)$. Note that there is exactly one out-edge at each vertex. The terms of cycle, length

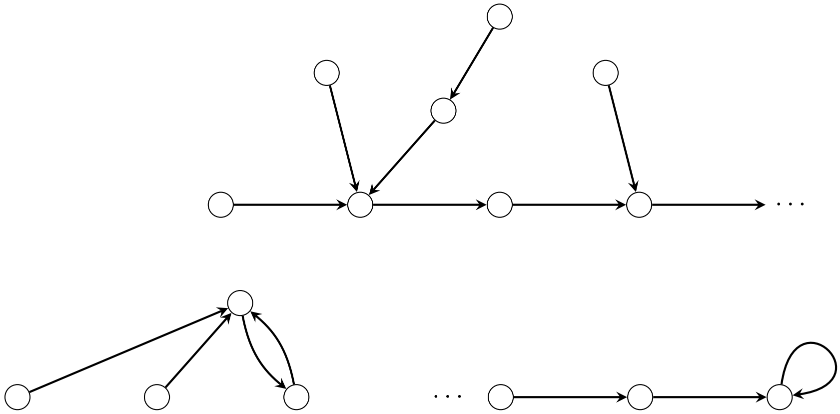

FIGURE 1. A mono-unary algebra

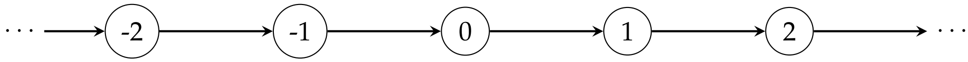

FIGURE 2. The algebra Z

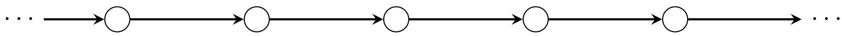

FIGURE 3. A line

of a cycle, cyclic element, connected monounary algebra are intuitively clear from a graph visualisation. Formal definitions can be found in [1, 9, 13].

Example 3.1. *A mono-unary algebra which is not connected is depicted in Figure 1. It consists of 3 (connected) components. Two components have a cycle, one is without a cycle. The component with 1-element cycle is infinite and this component is a mono-unary algebra with infinitely many different subalgebras.*

We denote by Z the mono-unary algebra that is defined on the set of all integer numbers with the successor function, see Figure 2.

A mono-unary algebra (A, h) is called a line if it is isomorphic to the algebra Z, see Figure 3.

The next assertion is obvious.

Lemma 3.4. *Let (A, h) be a mono-unary algebra. Then the following statements are equivalent:*

(1) (A, h) is a line,
(2) (A, h) is connected, A is infinite and the operation h is bijective,
(3) (A, h) is connected without a cycle and the operation h is bijective.

Lemma 3.5. *Let (A, h) be a connected mono-unary algebra. If (A, h) contains a subalgebra C such that C is a cycle or C is a line, then C is a retract of (A, h).*

Proof. Put $\varphi(c) = c$ for $c \in C$. Let $a \in A \setminus C$. Then there is $n \in \mathbb{N}$ such that $h^n(a) \in C$ and $h^{n-1}(a) \notin C$. The set C contains exactly one element b such that $h^n(b) = h^n(a)$. Put $\varphi(a) = b$. We have that $\varphi(A) = C$ and φ is the retraction. □

Now we come to the essential assignment in this section. Let I be a nonempty set. For each $i \in I$ let (B_i, h) be a mono-unary algebra. We denote by $\sum_{i \in I}(B_i, h)$ a mono-unary algebra

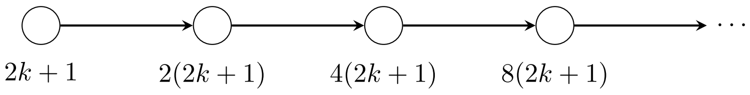

FIGURE 4. Infinite components of $(\mathbb{Z}, f)$, $k \in \mathbb{Z}$

which is a disjoint union of algebras (B_i, h), $i \in I$. Let $(A, h) = \sum_{i \in I}(B_i, h)$ and (B_i, h) be connected for all $i \in I$. Let $i \in I$. If (B_i, h) contains a cycle of length k for some $k \in \mathbb{N}$, then we denote by (C_i, h) a cycle of length k. Else we denote by (C_i, h) a line. Put

$$(A, h)^\diamond = \sum_{i \in I}(C_i, h).$$

If $\mathcal{A} = (A, h)$ and $f = h$ in Lemma 3.3, then $\overline{(A, h)} \cong (A, h)^\diamond$ according to [3], Lemma 4.

Theorem 3.1. *Let $\mathcal{A} = (A, F)$ and $f \in T(F)$ be unary term such that f is an endomorphism of the algebra $\mathcal{A}$. Suppose that $\{\mathbb{N}, \mathcal{A}_i, \varphi_{i,j}\}$ is $\mathcal{A}$-uniform direct system from Lemma 3.3 and $\{\mathbb{N}, \mathcal{A}_i, \varphi_{i,j}\} \longrightarrow \overline{\mathcal{A}} = (\overline{A}, F)$. Then $(\overline{A}, f) \cong (A, f)^\diamond$.*

If $\mathcal{A}$ is an algebra with easy direct limits, then the mono-unary algebra $(A, f)^\diamond$ is isomorphic to a retract of (A, f).

Proof. Obviously $\{\mathbb{N}, (A_i, f), \varphi_{i,j}\}$ is (A, f)-uniform direct system of mono-unary algebras and $\{\mathbb{N}, (A_i, f), \varphi_{i,j}\} \longrightarrow (\overline{A}, f)$. In view of Lemma 4 of [3] is $(\overline{A}, f) \cong (A, f)^\diamond$.

Suppose that $\mathcal{A} = (A, F)$ is an algebra with easy direct limits. Then $\overline{\mathcal{A}}$ is isomorphic to a retract of $\mathcal{A}$. It yields that $(\overline{A}, f)$ is isomorphic to a retract of (A, f) according to Lemma 3.2. □

Corollary 3.1. *Let $\mathcal{A} = (A, F)$ and $f \in T(F)$ be unary term such that f is an endomorphism of the algebra $\mathcal{A}$. Then*

(1) *there exists $\mathcal{B} = (B, F) \in \underrightarrow{\mathbf{L}}\mathcal{A}$ such that f is bijective on B,*
(2) *if $\mathcal{A}$ is an algebra with easy direct limits, then there exists $\mathcal{B} = (B, F) \in \mathbf{R}\mathcal{A}$ such that f is bijective on B.*

Example 3.2. *The operation $f(x) = x + x$ is a term of $\{+\}$, where $+$ is usual binary addition operation. The additive group of integers $(\mathbb{Z}, +, -, 0)$ is commutative and therefore f is an endomorphism of this group. Let us look at the mono-unary algebra $(\mathbb{Z}, f)$. It consists of infinitely many components. One of them is $\{0\}$, others are infinite and isomorphic to each other, see Figure 4. Every infinite component is generated by an odd number, therefore no subalgebra of $(\mathbb{Z}, f)$ contains a line.*

The algebra $(\mathbb{Z}, f)^\diamond$ consists of one 1-element cycle and infinitely many lines according to the definition. Thus $(\mathbb{Z}, f)^\diamond$ is not isomorphic to a retract of $(\mathbb{Z}, f)$. Theorem 3.1 implies that the group $(\mathbb{Z}, +, -, 0)$ is not an algebra with easy direct limits.

Example 3.3. *The operation $g(x) = x \cdot x$ is a term of $\{\cdot\}$, where $\cdot$ is usual binary multiplication operation.*

The multiplicative monoid of real numbers $(\mathbb{R}, \cdot, 1)$ is commutative and therefore g is an endomorphism of this group at the same time. The mono-unary algebra $(\mathbb{R}, g)$ consists of uncountable many components. Finite ones are $\{0\}$ and $\{-1, 1\}$, others are mutually isomorphic, see Figure 5. Every infinite component contains one line as its subalgebra. The algebra $(\mathbb{R}, g)^\diamond$ consists of two 1-element cycles and lines. (Note that it is isomorphic to $(\mathbb{R}_0^+, g)$, where $\mathbb{R}_0^+$ is the set of all non-negative real numbers.) Thus $(\mathbb{R}, g)^\diamond$ is isomorphic to a retract of $(\mathbb{R}, g)$ according to Lemma 3.5. Therefore Theorem 3.1 does not give the answer whether $(\mathbb{R}, \cdot, 1)$ is the algebra with easy direct limits.

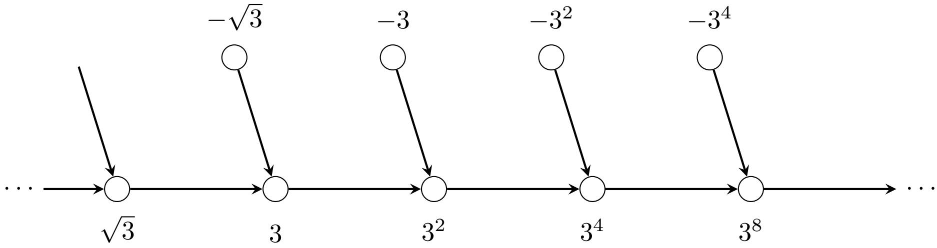

FIGURE 5. The component of $(\mathbb{R}, g)$ which contains the number 3

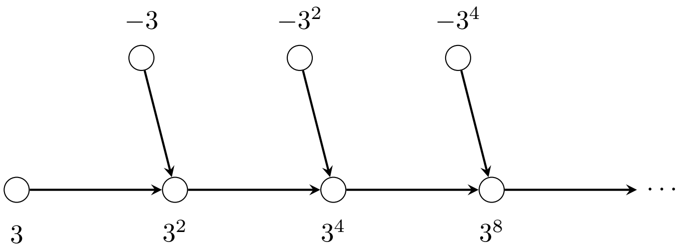

FIGURE 6. The component of $(\mathbb{Q}, g)$ which contains the number 3

Consider the monoid $(\mathbb{Q}, \cdot, 1)$. The mono-unary algebra $(\mathbb{Q}, g)$ consists of infinitely many components too. Infinite ones are isomorphic to each other, see Figure 6. There is no line as a subalgebra. The algebra $(\mathbb{Q}, g)^\diamond$ is not isomorphic to a retract of $(\mathbb{Q}, g)$. (In fact, the $(\mathbb{Q} \setminus \{0\}, g)^\diamond$ is isomorphic to the algebra $(\mathbb{Z}, f)^\diamond$ from the previous example.) Therefore $(\mathbb{Q}, \cdot, 1)$ is not the algebra with easy direct limits according to Theorem 3.1.

4. UNARY TERMS WHICH ARE ENDOMORPHISMS

Let $\mathcal{A} = (A, F)$ be an algebra. The identity mapping is a term operation, since it is a projection. It is an endomorphism of $\mathcal{A}$ at the same time. If $\varphi, \psi \in T(F)$ are endomorphisms of $\mathcal{A}$, then obviously $\varphi \circ \psi \in T(F)$ and $\varphi \circ \psi$ is an endomorphism of $\mathcal{A}$. We will see that

- all unary term operations are endomorphisms of $\mathcal{A}$ in the case of abelian groups and mono-unary algebras;
- the identity mapping is the only term operation that is an endomorphism of $\mathcal{A}$ in the case of rings of characteristic zero with 1;
- unary term operations which are endomorphisms of $\mathcal{A}$ are closely linked to the structure of $\mathcal{A}$ in the case of unary algebras.

Proposition 4.1. *Let $(G, +, -, 0)$ be an abelian group. Then every unary term is an endomorphism of $(G, +, -, 0)$.*

Proof. Every unary term of an additive group has a form $f(x) = kx$, for some $k \in \mathbb{Z}$. Therefore $f(0) = 0$ for $0 \in G$, Assume that $g_1, g_2 \in G$. Then

$$f(-g_1) = k(-g_1) = -(kg_1) = -f(g_1)$$

according to group properties. Further,

$$f(g_1 + g_2) = k(g_1 + g_2) = kg_1 + kg_2$$

since $(G,+,-,0)$ is abelian. □

Proposition 4.2. *Let* $(R,+,-,\cdot,0,1)$ *be a ring of characteristic zero with* 1. *Then a unary term is an endomorphism of* $(R,+,-,,\cdot,0,1)$ *if and only if it is the identity operation.*

Proof. It is obvious that every unary term of a ring has a form

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0,$$

where $a_i \in \mathbb{Z}, i \in \{1,\ldots,n\}$. Every ring with characteristic zero contains a subalgebra which is isomorphic to the ring of integers $(\mathbb{Z},+,\cdot,-,0,1)$. Without loss of generality suppose that $\mathbb{Z} \subseteq R$.

Assume that f is an endomorphism of $(R,+,\cdot,-,0,1)$. Hence f is an endomorphism of the group $(R,+,-,0)$ and $f(1)=1$. Thus $f(m)=m$ is valid for every $m \in \mathbb{Z}$. Therefore $a_0 = 0$.

We obtained

$$\begin{aligned} a_0 &= f(0) = 0,\\ a_n + a_{n-1} + \ldots + a_1 &= f(1) = 1,\\ a_n 2^n + a_{n-1}2^{n-1} + \ldots + a_1 2 &= f(2) = 2,\\ &\vdots\\ a_n n^n + a^{n-1}n^{n-1} + \ldots + a_1 n &= f(n) = n. \end{aligned}$$

Dividing $k-$th equation by k for each $k \in \{1,\ldots,n\}$, we obtain a system of linear equations:

$$\text{(4.2)} \qquad \left\{ \begin{array}{ccccccccc} a_n & + & a_{n-1} & + & \ldots & + & a_2 & + & a_1 = 1\\ a_n 2^{n-1} & + & a_{n-1}2^{n-2} & + & \ldots & + & a_2 2 & + & a_1 = 1\\ \vdots & & & & & & & & \\ a_n n^{n-1} & + & a_{n-1}n^{n-2} & + & \ldots & + & a_2 n & + & a_1 = 1 \end{array} \right. .$$

The coefficient matrix has the form:

$$\begin{bmatrix} 1 & 1 & \ldots & 1 & 1\\ 2^{n-1} & 2^{n-2} & \ldots & 2 & 1\\ \vdots & & & & \\ n^{n-1} & n^{n-2} & \ldots & n & 1 \end{bmatrix}.$$

This is a Vandermond matrix with non-zero determinant, which implies that the system of linear equations (4.2) has exactly one solution. Hence,

$$a_1 = 1, a_2 = \ldots = a_n = 0$$

is the only solution. This completes the proof. □

Proposition 4.3. *Let* (A,h) *be a mono-unary algebra. Then every unary term is an endomorphism of* (A,h).

Proof. Let g be a unary term over F. Then $g = h^k$ for some $k \in \mathbb{N}_0$. Assume that $a \in A$. We have

$$g(h(a)) = h^k(h(a)) = h^{k+1}(a) = h(h^k(a)) = h(g(a)).$$

□

Next example demonstrates that there are algebras which have a unary term that is not an endomorphism and at the same time they have at least two unary terms that are endomorphisms.

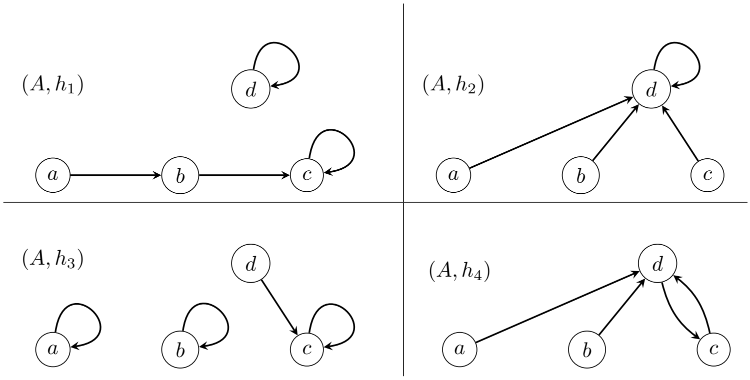

FIGURE 7. Mono-unary algebras from Example 4.4

Example 4.4. *Let $A = \{a, b, c, d\}$ and unary operations $h_1 - h_4$ on A are given by Table 1.*

	h_1	h_2	h_3	h_4
a	b	d	a	d
b	c	d	b	d
c	c	d	c	d
d	d	d	c	c

TABLE 1. Operations $h_1 - h_4$

Consider unary algebra $\mathcal{A} = (A, \{h_1, h_2, h_3, h_4\})$. We name three unary terms different from the identity that are endomorphisms of $\mathcal{A}$ and three unary terms that are not endomorphisms of $\mathcal{A}$. The term h_1 is an endomorphism of $\mathcal{A}$, since it is an endomorphism of mono-unary algebras (A, h_i) for $i = 1, 2, 3, 4$; algebras (A, h_i) are in Figure 7. The term h_1^2 is an endomorphism of $\mathcal{A}$ since the composition of endomorphisms is an endomorphism. Note that for $k \in \mathbb{N}, k > 2$ is $h_1^k = h_1^2$ on A. Further, it is easy to check that h_4^2 is an endomorphism of (A, h_i) for $i = 1, 2, 3, 4$ and therefore this term is an endomorphism of $\mathcal{A}$. Terms h_2, h_3, h_4 are not endomorphisms of $\mathcal{A}$.

The following assertion characterizes all unary algebras which have the property that all unary terms are endomorphisms at the same time. It follows from definitions.

Proposition 4.4. *Let $\mathcal{A} = (A, F)$ be a unary algebra. Then $\varphi : A \to A$ is an endomorphism of $\mathcal{A}$ if and only if it is an endomorphism of the mono-unary algebra (A, h) for each $h \in F$. Further, the following properties are equivalent:*

(1) every unary term of $\mathcal{A}$ is an endomorphism of $\mathcal{A}$,
(2) if $h \in F$, then h is an endomorphism of $\mathcal{A}$.

The last statement helps recognise unary algebras which have a constant term operation that is an endomorphism at the same time.

Proposition 4.5. *Let (A, F) be a unary algebra and $F = \{h_i, i \in I\}$, where h_i is a unary operation for each $i \in I$.*
Suppose that $g \in T(F)$ is unary and $a^\star \in A$ are such that

(1) $h_i(a^\star) = a^\star$ for each $i \in I$,
(2) there exists $k \in \mathbb{N}$ such that $g^k(a) = a^\star$ for each $a \in A$.

Then $g^k \in T(F)$ and g^k is an endomorphism of (A, F).

Proof. The condition (2) says that the mapping g^k is constant on A. We have

$$g^k(h_i(a)) = a^\star = h_i(a^\star) = h_i(g^k(a)).$$

□

Example 4.5. *Consider the unary algebra $(A, \{h_1, h_3, h_4^2\})$, where A, h_1, h_3, h_4 are from the previous example. We denote by g the constant operation equal to c. Then $g = h_3 \circ h_4^2 \in T(\{h_1, h_3, h_4^2\})$. If we put $a^\star = c$ and $k = 1$, then suppositions of Proposition 4.5 are valid. Thus g is an endomorphism of $(A, \{h_1, h_3, h_4^2\})$.*

Acknowledgments

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Research Article

Lebesgue points and summability of higher dimensional Fourier transforms

FERENC WEISZ*

ABSTRACT. The well known Lebesgue's theorem about the almost everywhere convergence of the one-dimensional Fejér means is generalized for five different, more general summability methods and for higher dimensional functions from the Wiener amalgam space $W(L_1, \ell_\infty)(\mathbb{R}^d)$.

Keywords: Fourier transforms, Fejér summability, θ-summability, Lebesgue points.

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1. INTRODUCTION

The classical theorem of Lebesgue [21] says that the Fejér means [11] of a one-dimensional integrable function f converge almost everywhere to f. More exactly,

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T s_t f(x)\, dt = f(x)$$

at each Lebesgue point of f, thus almost everywhere, where $s_t f$ denotes the tth Dirichlet integral of the one-dimensional function f. In this survey paper, we generalize this theorem for higher dimensional functions and for more general summability methods. We give five different types of summability methods and five generalizations. All the five summability methods are investigated exhaustively in the literature.

A general method of summation, the so called θ-summation method, which is generated by a single function θ and which includes the well known Fejér, Riesz, Weierstrass, Abel, etc. summability methods, is studied intensively in the literature (see e.g. Butzer and Nessel [6], Trigub and Belinsky [2, 33, 34], Liflyand [23], Gát [12, 13, 14], Goginava [15, 16, 17], Simon [29], Persson, Tephnadze and Wall [27] and Weisz [35, 44]). Lebesgue points of multi-dimensional functions are investigated in Belinsky, Liflyand and Trigub [3, 1] and in Feichtinger and Weisz [9, 10].

For higher dimensional functions the θ-summability can be defined by

$$\sigma_T^{q,\theta} f(x) := \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \theta\left(\frac{\|t\|_q}{T}\right) \widehat{f}(t) e^{\imath x\cdot t}\, dt \qquad (x\in\mathbb{R}^d, T>0)$$

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or by

$$\sigma_T^\theta f(x) := \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \prod_{j=1}^d \theta_j \left(\frac{|t_j|}{T_j} \right) \widehat{f}(t) e^{\imath x \cdot t} \, dt \qquad (x \in \mathbb{R}^d, T \in \mathbb{R}_+^d).$$

The special cases $q = 1, 2, \infty$ are investigated exhaustively in the literature. In particular, the case $q = 2$ in Stein and Weiss [31], Davis and Chang [8] and Grafakos [18], the case $q = 1$ in Berens [4, 5], Szili and Vértesi [32] and the case $q = \infty$ in Marcinkiewicz [24], Zhizhiashvili [45] and Weisz [35, 44]. The second type of summation was considered e.g. in Zygmund [46], Gát [12] and Weisz [35, 44].

In this paper, we introduce a different concept of Lebesgue points and a different Hardy-Littlewood maximal function for each summability just mentioned. We generalize Lebesgue's theorem for these new Lebesgue points and for the different summability methods and for higher dimensional functions from the Wiener amalgam space $W(L_1, \ell_\infty)(\mathbb{R}^d) \supset L_1(\mathbb{R}^d)$. All the results of this paper hold e.g. for the Weierstrass, Abel, Picard, Bessel, Fejér, de La Vallée-Poussin, Rogosinski and Riesz summations. This paper was the base of my talk given at the *3rd International Conference: Constructive Mathematical Analysis, July, 2025 in Konya (Türkiye).*

2. The one-dimensional θ-summability and Lebesgue's theorem

For a fixed $d \in \mathbb{N}$, $d \geq 2$, we equip the space $L_p(\mathbb{R}^d)$ with the norm

$$\|f\|_p := \begin{cases} \left(\int_{\mathbb{R}^d} |f|^p \, d\lambda \right)^{1/p}, & 0 < p < \infty; \\ \sup_{\mathbb{R}^d} |f|, & p = \infty. \end{cases}$$

The Fourier transform of a one-dimensional function $f \in L_1(\mathbb{R})$ is given by

$$\widehat{f}(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(t) e^{-\imath x t} \, dt \qquad (x \in \mathbb{R}),$$

where $\imath = \sqrt{-1}$. If $f \in L_p(\mathbb{R})$ for some $1 \leq p \leq 2$ and $f \in L_1(\mathbb{R})$, then the Fourier inversion formula holds:

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \widehat{f}(t) e^{\imath x t} \, dt \qquad (x \in \mathbb{R}). \tag{2.1}$$

The integrability condition of $\widehat{f}$ is a very strong condition. Without this condition, we may consider the Dirichlet integral $s_T f$:

$$s_T f(x) := \frac{1}{\sqrt{2\pi}} \int_{-T}^{T} \widehat{f}(t) e^{\imath x t} \, dt,$$

which is well defined. It is known that for $f \in L_p(\mathbb{R})$, $1 < p < \infty$,

$$\lim_{T \to \infty} s_T f = f \qquad \text{in the } L_p(\mathbb{R})\text{-norm and a.e.}$$

The norm convergence is due to Riesz [28] and the almost everywhere convergence is the famous theorem due to Carleson and Hunt (see Carleson [7] and Hunt [19] or recently Grafakos [18]).

This convergence does not hold for $p = 1$. However, using a summability method, we can generalize these results. The most known summability method is the Fejér method. The Fejér means are defined by

$$\sigma_T f(x) := \frac{1}{\sqrt{2\pi}} \int_{-T}^{T} \left(1 - \frac{|t|}{T} \right) \widehat{f}(t) e^{\imath x t} \, dt = \frac{1}{T} \int_0^T s_t f(x) \, dt.$$

We generalize this summation and introduce a very general summability method, the so called θ-summation defined by a function $\theta : \mathbb{R}_+ \to \mathbb{R}$ satisfying $\theta(0) = 1$. This summation contains all well known summability methods, such as the well known Weierstrass, Abel, Picard, Bessel, Fejér, de La Vallée-Poussin, Rogosinski and Riesz summations. Similarly to the Fejér means, in the Fourier inversion formula (2.1), we multiply the integrand by a suitable function θ . More precisely, let

$$\sigma_T^\theta f(x) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \theta\left(\frac{|t|}{T}\right) \widehat{f}(t) e^{\imath xt}\, dt. \tag{2.2}$$

It is easy to see that we get back the Fejér means if $\theta(t) = \max((1-|t|), 0)$. This definition can easily be extended to all $f \in W(L_1, \ell_\infty)(\mathbb{R})$.

It is know that the Fejér means converge to f as $T \to \infty$ when $f \in L_1(\mathbb{R})$. However, we can characterize the set of convergence as follows. The Hardy-Littlewood maximal function is defined by

$$Mf(x) := \sup_{h>0} \frac{1}{2h} \int_{-h}^{h} |f(x-t)|\, dt$$

and the following result holds (see e.g. Stein [30]):

Theorem 2.1. *If $f \in L_1(\mathbb{R})$, then*

$$\sup_{\rho>0} \rho\lambda(Mf > \rho) \le C\|f\|_1.$$

Moreover, if $1 < p \le \infty$ and $f \in L_p(\mathbb{R})$, then

$$\|Mf\|_p \le C_p \|f\|_p .$$

The weak type inequality of the theorem implies the next result due to Lebesgue.

Corollary 2.1. *If $f \in L_1(\mathbb{R})$, then*

$$\lim_{h\to 0} \frac{1}{2h} \int_{-h}^{h} f(x-t)\, dt = f(x) \qquad a.e.\ x \in \mathbb{R}.$$

This convergence is equivalent to the next two convergences:

$$\lim_{h\to 0} \frac{1}{2h} \int_{-h}^{h} f(x-t) - f(x)\, dt = 0 \qquad \text{a.e.}$$

and

$$\lim_{h\to 0} \frac{1}{2h} \left| \int_{-h}^{h} f(x-t) - f(x)\, dt \right| = 0 \qquad \text{a.e.}\,.$$

If we can go with the absolute value inside the integral, then we say that x is a Lebesgue point of f. More exactly, a point $x \in \mathbb{R}$ is called a Lebesgue point of $f \in L_1(\mathbb{R})$ if

$$\lim_{h\to 0} \frac{1}{2h} \int_{-h}^{h} |f(x-t) - f(x)|\, dt = 0.$$

We can see, that the Lebesgue points are depending only on f and if f is continuous at x, then x is a Lebesgue point of f. All our generalizations will have these properties.

Theorem 2.2. *Almost every point $x \in \mathbb{R}$ is a Lebesgue point of $f \in L_1(\mathbb{R})$.*

The next well known theorem was proved by Lebesgue [21] in 1905 (see also [11]).

Theorem 2.3. *For all Lebesgue points of $f \in L_1(\mathbb{R})$,*

$$\lim_{T\to\infty} \sigma_T f(x) = f(x).$$

In what follows, we will generalize this theorem for multi-dimensional functions and for the general θ-summability method.

3. The multi-dimensional θ-summability

Let us turn to the higher dimensional functions. For $t, x \in \mathbb{R}^d$, let

$$t \cdot x := \sum_{k=1}^d t_k x_k, \qquad \|x\|_q := \begin{cases} \left(\sum_{k=1}^d |x_k|^q\right)^{1/q}, & 0 < q < \infty; \\ \sup_{i=1,\ldots,d} |x_i|, & q = \infty. \end{cases}$$

The Fourier transform of a higher dimensional function $f \in L_1(\mathbb{R}^d)$ is given by

$$\widehat{f}(x) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} f(t) e^{-\imath x\cdot t}\, dt \qquad (x \in \mathbb{R}^d).$$

The inversion formula (2.1) holds in this case, too.

The first question is, how can we generalize the definition (2.2) for higher dimensional functions? There are some generalizations, which are investigated exhaustively in the literature. In the first natural generalization, instead of $|t|$ we write the q-norm $\|t\|_q$ of the d-dimensional vector $t = (t_1, \ldots, t_d)$ and the function θ remains a one-dimensional function. These means are called ℓ_q-θ-means. More exactly, suppose that $\theta : \mathbb{R}_+ \to \mathbb{R}$ satisfying

$$\theta \in C_0(\mathbb{R}_+), \qquad \theta(\|\cdot\|_q) \in L_1(\mathbb{R}^d), \qquad \theta(0) = 1. \tag{3.3}$$

Here $C_0(\mathbb{R}_+)$ denotes the set of continuous functions vanishing at infinity. The Tth ℓ_q-θ-mean of the function $f \in L_p(\mathbb{R}^d)$ $(1 \le p \le 2)$ is defined by

$$\sigma_T^{q,\theta} f(x) := \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \theta\left(\frac{\|t\|_q}{T}\right) \widehat{f}(t) e^{\imath x\cdot t}\, dt \qquad (x \in \mathbb{R}^d, T > 0).$$

Note that the integral is well defined and T is a positive real number. Usually three subcases are investigated in the literature: the ℓ_q-θ-means are called triangular if $q = 1$, circular if $q = 2$ and cubic if $q = \infty$ (see Figure 1). The cubic summability (when $q = \infty$) is also called Marcinkiewicz summability.

In the other natural generalization, instead of the function θ, we use d one-dimensional functions $\theta_1, \ldots, \theta_d$. The Tth rectangular θ-mean of the function $f \in L_p(\mathbb{R}^d)$ $(1 \le p \le 2)$ is given by

$$\sigma_T^{\theta} f(x) := \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \prod_{j=1}^d \theta_j\left(\frac{|t_j|}{T_j}\right) \widehat{f}(t) e^{\imath x\cdot t}\, dt \qquad (x \in \mathbb{R}^d, T \in \mathbb{R}_+^d).$$

In this case, we will always assume that

$$\theta = \theta_1 \otimes \cdots \otimes \theta_d, \qquad \theta_j \in L_1(\mathbb{R}_+) \cap C_0(\mathbb{R}_+) \qquad \text{and} \qquad \theta_j(0) = 1 \tag{3.4}$$

for all $j = 1, \ldots, d$. In this definition, $T = (T_1, \ldots, T_d) \in \mathbb{R}_+^d$ is a vector. Two subcases of this summability will be investigated, the restricted (when $T \in \mathbb{R}_+^d$ is in a cone) and the unrestricted (when $T \in \mathbb{R}_+^d$) summability. For each of the five generalizations, we need a new concept of Lebesgue points. The proofs are strongly different for different cases.

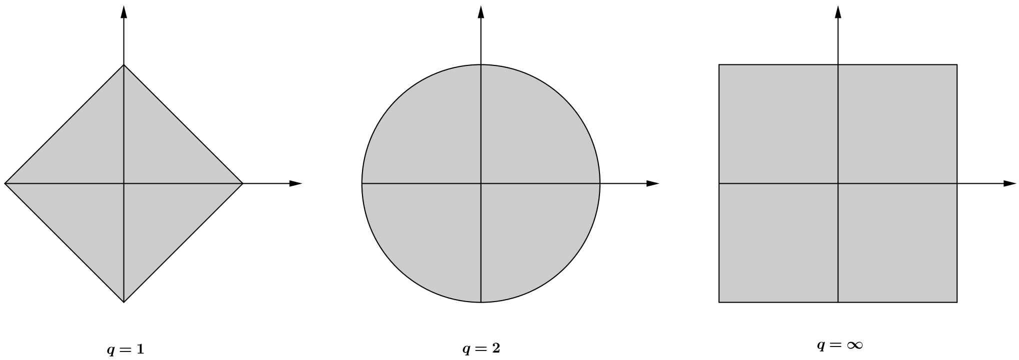

FIGURE 1. Regions of the ℓ_q-partial sums for $d = 2$.

4. Norm convergence of the higher dimensional summability means

In this section, we deal with the norm convergence of the ℓ_q-summability means and the rectangular means. First, we introduce the next function space. A Banach space B consisting of Lebesgue measurable functions on $\mathbb{R}^d$ is called a homogeneous Banach space if

(a) for all $f \in B$ and $x \in \mathbb{R}^d$, $T_x f \in B$ and $\|T_x f\|_B \sim \|f\|_B$,
(b) the function $x \mapsto T_x f$ from $\mathbb{R}^d$ to B is continuous for all $f \in B$,
(c) the functions in B are uniformly locally integrable, i.e., for every compact set $K \subset \mathbb{R}^d$ there exists a constant C_K such that

$$\int_K |f|\, d\lambda \le C_K \|f\|_B \qquad (f \in B).$$

Note that if B is a homogeneous Banach space on $\mathbb{R}$, then

$$B \subset W(L_1, \ell_\infty)(\mathbb{R})$$

(see Katznelson [20]). It is easy to see that the spaces $C_0(\mathbb{R}^d)$, $L_p(\mathbb{R}^d)$, $W(L_p, \ell_q)(\mathbb{R}^d)$, $W(L_p, c_0)(\mathbb{R}^d)$, $W(C, \ell_q)(\mathbb{R}^d)$, $W_I(L_p, c_0)(\mathbb{R}^d)$ $(1 \le p, q < \infty)$ and Lorentz spaces $L_{p,q}(\mathbb{R}^d)$ $(1 < p < \infty, 1 \le q < \infty)$ are all homogeneous Banach spaces. Note that the definition of Wiener amalgam spaces can be found in Section 5. Moreover, the space $C_u(\mathbb{R}^d)$ of uniformly continuous bounded functions endowed with the supremum norm is also a homogeneous Banach space.

4.1. Norm convergence of the ℓ_q-summability means.

For an integrable function f,

$$\sigma_T^{q,\theta} f(x) = \int_{\mathbb{R}^d} f(x-t) K_T^{q,\theta}(t)\, dt = f * K_T^{q,\theta}(x) \qquad (x \in \mathbb{R}^d, T > 0),$$

where the Tth ℓ_q-θ-kernel is given by

$$K_T^{q,\theta}(x) := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \theta\left(\frac{\|t\|_q}{T}\right) e^{\imath x \cdot t}\, dt = \frac{1}{(2\pi)^{d/2}} T^d \widehat{\theta_0}(Tx).$$

This definition can easily be extended to all $f \in W(L_1, \ell_\infty)(\mathbb{R}^d)$.

For $q=2$, we suppose in addition to (3.3) that

$$\widehat{\theta_0} \in L_1(\mathbb{R}^d), \tag{4.5}$$

where $\theta_0(x) := \theta(\|x\|_q)$. Since this condition is hard to check for $q=1$ and $q=\infty$, we use another concept in this case. Namely, we suppose that θ is continuous on $\mathbb{R}_+$, the support of θ is $[0,c]$ for some $0<c\leq\infty$ and θ is differentiable on $(0,c)$. Suppose further that

$$\theta(0)=1, \qquad \int_0^\infty \max\{t,1\}^d |\theta'(t)|\,dt < \infty, \qquad \lim_{t\to\infty} t^d\theta(t)=0 \tag{4.6}$$

and assume that

$$\left| \int_0^\infty \theta'(t)\, t \,\mathrm{soc}\,(tu)\,dt \right| \leq C u^{-\alpha} \tag{4.7}$$

for all $u>0$ and for some $0<\alpha<\infty$, where

$$\mathrm{soc}\, t := \begin{cases} \cos t, & \text{if } d \text{ is even;} \\ \sin t, & \text{if } d \text{ is odd.} \end{cases}$$

We can show that (4.6) implies that $\theta(\|\cdot\|_q) \in L_1(\mathbb{R}^d)$ for $q=1$ and $q=\infty$.

The next two theorems were proved in Berens, Li and Xu [4], Oswald [26] and Weisz [38, 36, 37]. Moreover, Li and Xu [22] investigated these theorem for Jacobi polynomials.

Theorem 4.4. *Under the conditions* (3.3) *and* (4.5) *with* $q=2$ *or* (4.6) *and* (4.7) *with* $q=1,\infty$, *we have*

$$\int_{\mathbb{R}^d} \left| K_T^{q,\theta}(x) \right| dx \leq C \qquad (T>0).$$

Theorem 4.5. *Assume that* B *is a homogeneous Banach space on* $\mathbb{R}^d$. *Under the conditions of Theorem 4.4,*

$$\left\| \sigma_T^{q,\theta} f \right\|_B \leq C \|f\|_B \qquad (T>0)$$

and

$$\lim_{n\to\infty} \sigma_T^{q,\theta} f = f \qquad \textit{in the } B\textit{-norm for all } f\in B.$$

Since $C_u(\mathbb{R}^d)$ is a homogeneous Banach space, we obtain:

Corollary 4.2. *If* f *is a uniformly continuous and bounded function, then, under the conditions of Theorem 4.4,*

$$\lim_{T\to\infty} \sigma_T^{q,\theta} f = f \qquad \textit{uniformly.}$$

4.2. **Norm convergence of the rectangular summability means.** For an integrable function f, the rectangular means can be written in the form

$$\sigma_T^\theta f(x) = \int_{\mathbb{R}^d} f(x-t) K_T^\theta(t)\,dt = f * K_T^\theta(x) \qquad (x\in\mathbb{R}^d, T\in\mathbb{R}_+^d),$$

where the Tth rectangular θ-kernel is given by

$$K_T^\theta(x) := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \prod_{j=1}^d \theta_j\left(\frac{|t_j|}{T_j}\right) e^{\imath x\cdot t}\,dt = \frac{1}{(2\pi)^{d/2}} \prod_{j=1}^d T_j \widehat{\theta_j}(T_j x_j).$$

If $\widehat{\theta_j} \in L_1(\mathbb{R})$ $(j=1,\dots,d)$, then we can extend the definition of the rectangular θ-means to all $f\in W(L_1,\ell_\infty)(\mathbb{R}^d)$.

Theorem 4.6. *Assume that B is a homogeneous Banach space on $\mathbb{R}^d$. If θ_j satisfies (3.4) and $\widehat{\theta_j} \in L_1(\mathbb{R})$ for all $j = 1, \ldots, d$, then*

$$\left\|\sigma_T^\theta f\right\|_B \leq C \left\|f\right\|_B \qquad (f \in B, T \in \mathbb{R}_+^d)$$

and

$$\lim_{T\to\infty} \sigma_T^\theta f = f \qquad \text{in the } B\text{-norm for all } f \in B.$$

As a consequence, we obtain:

Corollary 4.3. *If f is a uniformly continuous and bounded function, θ_j satisfies (3.4) and $\widehat{\theta_j} \in L_1(\mathbb{R})$ for all $j = 1, \ldots, d$, then*

$$\lim_{T\to\infty} \sigma_T^\theta f = f \qquad \text{uniformly.}$$

5. Almost everywhere convergence of the higher dimensional summability means

As mentioned earlier, we will consider three subcases of the ℓ_0-summability, $q = 2$, $q = \infty$ and $q = 1$. Moreover, we also study two subcases of the rectangular summability, the restricted and the unrestricted summability. We will introduce for each case a new Hardy-Littlewood maximal function and a new type of Lebesgue points.

First, we generalize the $L_p(\mathbb{R}^d)$ spaces and introduce new function spaces. A measurable function f belongs to the Wiener amalgam space $W(L_p, \ell_q)(\mathbb{R}^d)$ $(1 \leq p \leq \infty)$ if

$$\|f\|_{W(L_p,\ell_q)} := \left(\sum_{k\in\mathbb{Z}^d} \|f(\cdot + k)\|_{L_p[0,1)^d}^q\right)^{1/q} < \infty$$

if $1 \leq q < \infty$ and

$$\|f\|_{W(L_p,\ell_\infty)} := \left(\sup_{n_i\in\mathbb{Z}, i=1,\ldots,d} \int_{n_1}^{n_1+1} \cdots \int_{n_d}^{n_d+1} |f(x)|^p \, dx\right)^{1/p}$$

if $q = \infty$. It is easy to see that $W(L_p, \ell_p)(\mathbb{R}^d) = L_p(\mathbb{R}^d)$ and

$$W(L_\infty, \ell_1)(\mathbb{R}^d) \subset L_p(\mathbb{R}^d) \subset W(L_1, \ell_\infty)(\mathbb{R}^d) \qquad (1 \leq p \leq \infty).$$

We modify slightly the definition of $W(L_p, \ell_\infty)(\mathbb{R}^d)$. If we change the integrals and the supremums in this expression, then we obtain the definition of the iterated Wiener amalgam spaces. In other words, a function f is in the iterated Wiener amalgam spaces $W_I(L_p, \ell_\infty)(\mathbb{R}^d)$ $(1 \leq p \leq \infty)$ if

$$\|f\|_{W_I(L_p,\ell_\infty)} := \sup_{(i_1,\ldots,i_d)} \left(\sup_{n_{i_1}\in\mathbb{Z}} \int_{n_{i_1}}^{n_{i_1}+1} \cdots \sup_{n_{i_d}\in\mathbb{Z}} \int_{n_{i_d}}^{n_{i_d}+1} |f(x)|^p \, dx_{i_d} \cdots dx_{i_1}\right)^{1/p}$$

is finite. If we replace $|f(x)|^p$ by $|f(x)|^p(\log^+ |f(x)|)^k$ in the previous integral, then we get the definition of $W_I(L_p(\log L)^k, \ell_\infty)(\mathbb{R}^d)$ $(k \in \mathbb{N})$. Moreover, f is in the set $L_p(\log L)^k(\mathbb{R}^d)$ $(1 \leq p < \infty)$ if

$$\|f\|_{L_p(\log L)^k} := \left(\int_{\mathbb{R}^d} |f|^p(\log^+ |f|)^k \, d\lambda\right)^{1/p} < \infty,$$

where $\log^+ u := \max(0, \log u)$. It is easy to see that

$$W(L_p, \ell_\infty)(\mathbb{R}^d) \supset W_I(L_p(\log L)^k, \ell_\infty)(\mathbb{R}^d) \supset L_p(\log L)^k(\mathbb{R}^d), L_r(\mathbb{R}^d)$$

for all $1 \le p < r \le \infty$. Note that the space $W_I(L_p(\log L)^k, \ell_\infty)(\mathbb{R}^d)$ does not contain $L_p(\mathbb{R}^d)$.

5.1. **Circular summability.** For the circular summability, we need a new function space, the so called Herz space. We denote by $B(c,h)$ $(c \in \mathbb{R}^d, h > 0)$ the ball $\{x \in \mathbb{R}^d : \|x-c\|_2 < h\}$. Let the dyadic coronas be defined by

$$Q_k := B(0,2^k) \setminus B(0,2^{k-1}) \quad (k>0), \qquad Q_0 := B(0,1).$$

The Herz space $E_q(\mathbb{R}^d)$ contains all measurable functions f for which

$$\|f\|_{E_q} := \sum_{k=0}^{\infty} 2^{dk(1-1/q)} \left\|f \mathbf{1}_{Q_k}\right\|_q < \infty.$$

Then obviously

$$L_1(\mathbb{R}^d) = E_1(\mathbb{R}^d) \supset E_q(\mathbb{R}^d) \supset E_{q'}(\mathbb{R}^d) \supset E_\infty(\mathbb{R}^d), \qquad 1 < q < q' < \infty.$$

Now we introduce a straightforward generalization of the Hardy-Littlewood maximal function by integrating on cubes:

$$M_p f(x) := \sup_{h>0} \left(\frac{1}{(2h)^d} \int_{-h}^{h} \cdots \int_{-h}^{h} |f(x-t)|^p \, dt \right)^{1/p}.$$

A similar result holds as in the one-dimensional case (see e.g. Stein [30] and Weisz [41]).

Theorem 5.7. *For $1 \le p < \infty$,*

$$\sup_{\rho>0} \rho \lambda(M_p f > \rho)^{1/p} \le C\|f\|_p \qquad (f \in L_p(\mathbb{R}^d)),$$

$$\|M_p f\|_r \le C_r \|f\|_r \qquad (f \in L_r(\mathbb{R}^d), p < r \le \infty)$$

and

$$\|M_p f\|_{W(L_{p,\infty},\ell_\infty)} \le C\|f\|_{W(L_p,\ell_\infty)} \qquad (f \in W(L_p,\ell_\infty)(\mathbb{R}^d)),$$

$$\|M_p f\|_{W(L_r,\ell_\infty)} \le C_r\|f\|_{W(L_r,\ell_\infty)} \qquad (f \in W(L_r,\ell_\infty)(\mathbb{R}^d), p < r \le \infty).$$

The third inequality of the theorem implies:

Corollary 5.4. *If $f \in W(L_1,\ell_\infty)(\mathbb{R}^d)$, then*

$$\lim_{h\to 0} \frac{1}{(2h)^d} \int_{-h}^{h} \cdots \int_{-h}^{h} f(x-t) \, dt = f(x) \qquad a.e.\ x \in \mathbb{R}^d.$$

In other words,

$$\lim_{h\to 0} \frac{1}{(2h)^d} \int_{-h}^{h} \cdots \int_{-h}^{h} f(x-t) - f(x) \, dt = 0 \qquad \text{a.e. } x \in \mathbb{R}^d.$$

A point $x \in \mathbb{R}^d$ is called a p-Lebesgue point of f $(1 \le p < \infty)$ if

$$\lim_{h\to 0} \left(\frac{1}{(2h)^d} \int_{-h}^{h} \cdots \int_{-h}^{h} |f(x-t) - f(x)|^p \, dt \right)^{1/p} = 0.$$

We can check easily that all r-Lebesgue points are p-Lebesgue points, whenever $p < r$. The definition of a p-Lebesgue point is depending only on f and every continuity point is a p-Lebesgue point. The following theorem can be found in Butzer and Nessel [6], Stein and Weiss [31] or Feichtinger and Weisz [9, 10].

Theorem 5.8. *Almost every point $x \in \mathbb{R}^d$ is a p-Lebesgue point of $f \in W(L_p, \ell_\infty)(\mathbb{R}^d)$ if $1 \le p < \infty$.*

Recall that $L_p(\mathbb{R}^d) \subset W(L_p, \ell_\infty)(\mathbb{R}^d)$. The first generalization of Lebesgue's theorem reads as follows.

Theorem 5.9. *Let $\theta_0 \in L_1(\mathbb{R}^d)$, $1 \le p < \infty$ and $1/p + 1/q = 1$. If $\widehat{\theta}_0 \in E_q(\mathbb{R}^d)$, then*

$$\lim_{T\to\infty} \sigma_T^{2,\theta} f(x) = f(x)$$

for all p-Lebesgue points of $f \in W(L_p, \ell_\infty)(\mathbb{R}^d)$.

The theorem is due to Feichtinger and Weisz [10]. Originally, it was proved for Riesz summation, for $p = 1$ and for integrable functions without using the Herz spaces in Stein and Weiss [31] or Butzer and Nessel [6]. We proved in [10] that the converse of the theorem holds also.

Theorem 5.10. *Suppose that $\theta_0 \in L_1(\mathbb{R}^d)$, $\widehat{\theta}_0 \in L_1(\mathbb{R}^d)$, $1 \le p < \infty$ and $1/p + 1/q = 1$. If*

$$\lim_{T\to\infty} \sigma_T^{2,\theta} f(x) = f(x)$$

for all p-Lebesgue points of $f \in L_p(\mathbb{R}^d)$, then $\widehat{\theta}_0 \in E_q(\mathbb{R}^d)$.

5.2. **Unrestricted rectangular summability.** Here we study another Hardy-Littlewood maximal function where we integrate on rectangles and not on cubes as before. The strong Hardy-Littlewood maximal function is defined by

$$M_{s,p} f(x) := \sup_{h_1,\ldots,h_d>0} \left(\frac{1}{2^d h_1 \cdots h_d} \int_{-h_1}^{h_1} \cdots \int_{-h_d}^{h_d} |f(x-t)|^p \, dt \right)^{1/p}.$$

The next theorem was proved by the author [41].

Theorem 5.11. *Assume that $1 \le p < \infty$ and $I = I_1 \times \cdots \times I_d$ with $|I_1| = \cdots = |I_d| = 1$. If $f \in L_p(\log L)^{d-1}(\mathbb{R}^d)$, then*

$$\sup_{\rho>0} \rho \lambda(x : M_{s,p} f(x) > \rho, x \in I)^{1/p} \le C_p + C_p \|f\|_{L_p(\log L)^{d-1}}.$$

For $p < r \le \infty$,

$$\|M_{s,p} f\|_r \le C_r \|f\|_r \qquad (f \in L_r(\mathbb{R}^d)).$$

If $f \in W_I(L_p(\log L)^{d-1}, \ell_\infty)(\mathbb{R}^d)$, then

$$\|M_{s,p} f\|_{W(L_{p,\infty}, \ell_\infty)} \le C_p + C_p \|f\|_{W_I(L_p(\log L)^{d-1}, \ell_\infty)}$$

and, for $p < r \le \infty$,

$$\|M_{s,p} f\|_{W(L_r, \ell_\infty)} \le C_r \|f\|_{W_I(L_r, \ell_\infty)} \qquad (f \in W_I(L_r, \ell_\infty)(\mathbb{R}^d)).$$

Corollary 5.5. *If $f \in W_I(L_1(\log L)^{d-1}, \ell_\infty)(\mathbb{R}^d)$, then*

$$\lim_{h\to 0} \frac{1}{\prod_{j=1}^d (2h_j)} \int_{-h_1}^{h_1} \ldots \int_{-h_d}^{h_d} f(x-t) \, dt = f(x) \qquad a.e.\ x \in \mathbb{R}^d.$$

A point $x \in \mathbb{T}^d$ is called a strong p-Lebesgue point of f $(1 \le p < \infty)$ if

$$\lim_{h\to 0} \left(\frac{1}{2^d h_1 \cdots h_d} \int_{-h_1}^{h_1} \cdots \int_{-h_d}^{h_d} |f(x-t) - f(x)|^p \, dt \right)^{1/p} = 0.$$

Here $h \to 0$ means that $h_i \to 0$ for all $i = 1, \ldots, d$. Again, every continuity point is a strong p-Lebesgue point.

The following two theorems are due to the author [39].

Theorem 5.12. *For $1 \leq p < \infty$, almost every point $x \in \mathbb{R}^d$ is a strong p-Lebesgue point of $f \in W_I(L_p(\log L)^{d-1}, \ell_\infty)(\mathbb{R}^d)$.*

Recall that $\theta = \theta_1 \otimes \cdots \otimes \theta_d$ satisfies (3.4). It is easy to see that $\widehat{\theta} \in E_q(\mathbb{R}^d)$ if and only if $\widehat{\theta_j} \in E_q(\mathbb{R})$ for all $j = 1, \ldots, d$.

Theorem 5.13. *Let $\theta \in L_1(\mathbb{R}^d)$, $1 \leq p < \infty, 1/p + 1/q = 1$ and $\widehat{\theta} \in E_q(\mathbb{R}^d)$. If $f \in W_I(L_p(\log L)^{d-1}, \ell_\infty)(\mathbb{R}^d)$, x is a strong p-Lebesgue point of f and $M_{s,p}f(x)$ is finite, then*

$$\lim_{T\to\infty} \sigma_T^\theta f(x) = f(x).$$

Obviously, the convergence holds almost everywhere. Here $T \to \infty$ means again that $T_j \to \infty$ for all $j = 1, \ldots, d$. The iterated Wiener amalgam space $W_I(L_p(\log L)^{d-1}, \ell_\infty)(\mathbb{R}^d)$ is the largest function space for which these two theorems hold, even if we consider only almost everywhere convergence. So they are not true either for $W(L_p, \ell_\infty)(\mathbb{R}^d)$ or for $L_p(\mathbb{R}^d)$ (see Gát [12]). Now we formulate the converse of the preceding theorem.

Theorem 5.14. *Suppose that $\theta \in L_1(\mathbb{R}^d)$, $\widehat{\theta} \in L_1(\mathbb{R}^d)$, $1 \leq p < \infty$ and $1/p + 1/q = 1$. If*

$$\lim_{T\to\infty} \sigma_T^\theta f(x) = f(x)$$

for all strong p-Lebesgue points of $f \in L_p(\mathbb{R}^d)$, then $\widehat{\theta} \in E_q(\mathbb{R}^d)$.

5.3. **Restricted rectangular summability.** The third generalization is almost the same as the second one, the difference is that we assume here that T is in a cone (see Figure 2). For a given $\tau \geq 1$, we define a cone by

$$\mathbb{R}_\tau^d := \{x \in \mathbb{R}_+^d : \tau^{-1} \leq x_i/x_j \leq \tau, i, j = 1, \ldots, d\}.$$

The choice $\omega = 1$ obviously yields the diagonal.

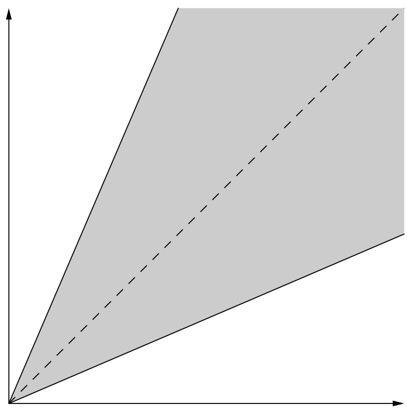

FIGURE 2. The cone for $d = 2$.

Instead of the Herz spaces we use here a weighted version of these spaces. The weighted Herz space $E_q^\omega(\mathbb{R})$ $(\omega \geq 0)$ contains all measurable functions f for which

$$\|f\|_{E_q^\omega} := \sum_{k=0}^{\infty} 2^{k(\omega+1-1/q)} \left\|f\mathbf{1}_{Q_k}\right\|_q < \infty.$$

Obviously,

$$E_q(\mathbb{R}) = E_q^0(\mathbb{R}) \supset E_q^\omega(\mathbb{R}) \qquad 0 \le \omega < \infty$$

and

$$L_1(\mathbb{R}) \supset E_1^\omega(\mathbb{R}) \supset E_q^\omega(\mathbb{R}) \supset E_{q'}^\omega(\mathbb{R}) \supset E_\infty^\omega(\mathbb{R}), \qquad 1 < q < q' < \infty.$$

Here, we cannot use the maximal functions introduced earlier, so we have to introduce a third generalization of the maximal function. For $\omega > 0$ and $1 \le p < \infty$, the Hardy-Littlewood maximal function $\mathcal{M}_p^{\omega,1} f$ is given by

$$\mathcal{M}_p^{\omega,1} f(x) := \sup_{i \in \mathbb{N}^d, h>0} 2^{-\omega \|i\|_1} \left(\frac{1}{(2h)^d 2^{\|i\|_1}} \int_{-2^{i_1}h}^{2^{i_1}h} \cdots \int_{-2^{i_d}h}^{2^{i_d}h} |f(x-t)|^p \, dt \right)^{1/p}.$$

If $\omega = 0$, we get back the definition of the strong Hardy-Littlewood maximal function $M_{s,p}f$. In contrary to the strong maximal function, due to the weight $2^{-\omega\|i\|_1}$, the weak type (p,p) inequality will be true for $\mathcal{M}_p^{\omega,1}$ (see Weisz [41]).

Theorem 5.15. *For $1 \le p < \infty$ and $\omega > 0$,*

$$\sup_{\rho>0} \rho \lambda(\mathcal{M}_p^{\omega,1} f > \rho)^{1/p} \le C\|f\|_p \qquad (f \in L_p(\mathbb{R}^d)),$$

$$\left\|\mathcal{M}_p^{\omega,1} f\right\|_r \le C_r \|f\|_r \qquad (f \in L_r(\mathbb{R}^d), p < r \le \infty)$$

and

$$\left\|\mathcal{M}_p^{\omega,1} f\right\|_{W(L_{p,\infty},\ell_\infty)} \le C\|f\|_{W(L_p,\ell_\infty)} \qquad (f \in W(L_p,\ell_\infty)(\mathbb{R}^d)),$$

$$\left\|\mathcal{M}_p^{\omega,1} f\right\|_{W(L_r,\ell_\infty)} \le C_r\|f\|_{W(L_r,\ell_\infty)} \qquad (f \in W(L_r,\ell_\infty)(\mathbb{R}^d), p < r \le \infty).$$

Note that the definitions of the p-Lebesgue points and strong p-Lebesgue points can be rewritten as

$$\lim_{r\to 0} \sup_{0<h<r} \left(\frac{1}{(2h)^d} \int_{-h}^{h} \cdots \int_{-h}^{h} |f(x-t) - f(x)|^p \, dt \right)^{1/p} = 0$$

and

$$\lim_{r\to 0} \sup_{0<h_j<r, j=1,\ldots,d} \left(\frac{1}{\prod_{j=1}^d (2h_j)} \int_{-h_1}^{h_1} \cdots \int_{-h_d}^{h_d} |f(x-t) - f(x)|^p \, dt \right)^{1/p} = 0,$$

respectively. Similarly to this definition, we introduce a new type of Lebesgue points. For $1 \le p < \infty$ and $\omega > 0$, a point $x \in \mathbb{R}^d$ is called a (p,ω)-Lebesgue point of f if

$$\lim_{r\to 0} \sup_{i \in \mathbb{N}^d, h>0, 2^{i_k}h<r, k=1,\ldots d} 2^{-\omega\|i\|_1} \left(\frac{1}{(2h)^d 2^{\|i\|_1}} \int_{-2^{i_1}h}^{2^{i_1}h} \cdots \int_{-2^{i_d}h}^{2^{i_d}h} |f(x-t) - f(x)|^p \, dt \right)^{1/p} = 0.$$

If $\omega = 0$, then the $(p,0)$-Lebesgue points are the same as the strong p-Lebesgue points. If f is continuous at x, then x is a (p,ω)-Lebesgue point of f for all $1 \le p < \infty$ and $\omega > 0$. The next two theorems are due to the author [42]. A first version of Theorem 5.17 was shown by Marcinkiewicz and Zygmund [25] in 1939. Later Gát [12] and the author [35, 44] proved the almost everywhere convergence.

Theorem 5.16. *For $1 \le p < \infty$, almost every point $x \in \mathbb{R}^d$ is a (p,ω)-Lebesgue point of $f \in W(L_p,\ell_\infty)(\mathbb{R}^d)$.*

Theorem 5.17. *Let $\theta_i \in L_1(\mathbb{R})$, $1 \le p < \infty$, $1/p + 1/q = 1$, $\omega > 0$ and $\widehat{\theta}_i \in E_q^\omega(\mathbb{R})$ $(i = 1, \ldots, d)$. If $f \in W(L_p, \ell_\infty)(\mathbb{R}^d)$, x is a (p,ω)-Lebesgue point of f and $\mathcal{M}_p^{\omega,1} f(x)$ is finite, then*

$$\lim_{T\to\infty, T\in\mathbb{R}_\tau^d} \sigma_T^\theta f(x) = f(x).$$

Of course, the convergence holds almost everywhere.

5.4. **Cubic summability.** In the fourth generalization, when $q = \infty$, we do not use Herz spaces. Instead, we will use conditions (4.6) and (4.7).

To introduce the next Hardy-Littlewood maximal function, let us denote by $P_{2^{i_1}h,\ldots,2^{i_d}h}$ a parallelepiped, whose center is the origin and whose sides are parallel to the axes and/or to the diagonals and whose kth side length is $2^{i_k+1}h$ if the kth side is parallel to an axis and $\sqrt{2}2^{i_k+1}h$ if the kth side is parallel to a diagonal $(i \in \mathbb{N}^d, h > 0, k = 1, \ldots, d)$. More exactly, at least one side of $P_{2^{i_1}h,\ldots,2^{i_d}h}$ is parallel to one of the axes and the other sides are parallel to the axes and/or to the diagonals.

For $\omega > 0$ and $1 \le p < \infty$, the Hardy-Littlewood maximal function $\mathcal{M}_p^\omega f$ is given by

$$\mathcal{M}_p^\omega f(x) := \sup_{P_{2^{i_1}h,\ldots,2^{i_d}h}, i\in\mathbb{N}^d, h>0} 2^{-\omega\|i\|_1} \left(\frac{1}{|P_{2^{i_1}h,\ldots,2^{i_d}h}|} \int_{P_{2^{i_1}h,\ldots,2^{i_d}h}} |f(x-t)|^p \, dt \right)^{1/p},$$

where the supremum is taken over all parallelepipeds $P_{2^{i_1}h,\ldots,2^{i_d}h}$ $(i \in \mathbb{N}^d, h > 0)$ just defined. If we take the supremum only over all rectangles with sides parallel to the axes, we get back the definition of the maximal operator $\mathcal{M}_p^{\omega,1} f$. 5.15 remains true for the maximal operator $\mathcal{M}_p^\omega$ if $1 \le p < \infty$ and $\omega > 0$.

For $1 \le p < \infty$ and $\omega > 0$, a point $x \in \mathbb{R}^d$ is called a strong (p,ω)-Lebesgue point of f if

$$\lim_{r\to 0} \sup_{\substack{P_{2^{i_1}h,\ldots,2^{i_d}h}, i\in\mathbb{N}^d, h>0 \\ 2^{i_k}h<r, k=1,\ldots d}} 2^{-\omega\|i\|_1} \left(\frac{1}{|P_{2^{i_1}h,\ldots,2^{i_d}h}|} \right)^{1/p} \left(\int_{P_{2^{i_1}h,\ldots,2^{i_d}h}} |f(x-t) - f(x)|^p \, dt \right)^{1/p} = 0.$$

The next theorems of this subsection were proved in [40].

Theorem 5.18. *Almost every point $x \in \mathbb{R}^d$ is a strong (p,ω)-Lebesgue point of $f \in W(L_p, \ell_\infty)(\mathbb{R}^d)$ if $1 \le p < \infty$.*

If $p > 1$ and $f \in W(L_p, \ell_\infty)(\mathbb{R}^d)$, then we do not need the concept of strong (p,ω)-Lebesgue points just introduced, it is enough to consider (p,ω)-Lebesgue points.

Theorem 5.19. *Suppose that $1 < p < \infty$ and the conditions (4.6) and (4.7) are satisfied. If $f \in W(L_p, \ell_\infty)(\mathbb{R}^d)$, x is a (p,ω)-Lebesgue point of f and $\mathcal{M}_p^{\omega,1} f(x)$ is finite, then*

$$\lim_{T\to\infty} \sigma_T^{\infty,\theta} f(x) = f(x).$$

This result does not hold for $p = 1$. In this case, we need the concept of strong (p,ω)-Lebesgue points.

Theorem 5.20. *Suppose that the conditions (4.6) and (4.7) are satisfied. If $f \in W(L_1, \ell_\infty)(\mathbb{R}^d)$, x is a strong $(1,\omega)$-Lebesgue point of f and $\mathcal{M}_1^\omega f(x)$ is finite, then*

$$\lim_{T\to\infty} \sigma_T^{\infty,\theta} f(x) = f(x).$$

Using the theorems of this section, we have given simple proofs for the classical strong summability results in [40].

5.5. **Triangular summability.** The convergence results are similar to those for cubic summability and are proved in [43]. We use the same Hardy-Littlewood maximal function and the same type of Lebesgue points as in Subsection 5.4.

Theorem 5.21. *Suppose that $1 < p < \infty$ and the conditions (4.6) and (4.7) are satisfied. If $f \in W(L_p, \ell_\infty)(\mathbb{R}^d)$, x is a (p,ω)-Lebesgue point of f and $\mathcal{M}_p^{\omega,1} f(x)$ is finite, then*

$$\lim_{T\to\infty} \sigma_T^{1,\theta} f(x) = f(x).$$

Theorem 5.22. *Suppose that the conditions (4.6) and (4.7) are satisfied. If $f \in W(L_1, \ell_\infty)(\mathbb{R}^d)$, x is a strong $(1,\omega)$-Lebesgue point of f and $\mathcal{M}_1^{\omega} f(x)$ is finite, then*

$$\lim_{T\to\infty} \sigma_T^{1,\theta} f(x) = f(x).$$

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Research Article

Wavelet-based approximation operators: applications to bivariate functions and digital image processing

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ABSTRACT. This work is a continuation of the author's very recent studies on the newly introduced wavelet based approximation operators, especially Bernstein operators [14]. The main goal of the present study is to construct and investigate bivariate case of these operators. In accordance with this purpose, we introduce two dimensional wavelet type Bernstein operators via wavelets and examine some characteristic properties together with their approximation results. Moreover, we give some application to bivariate functions and digital image processing.

Keywords: Bivariate Bernstein operators, wavelets, compactly supported Daubechies wavelets, image processing.

2020 Mathematics Subject Classification: 42C40, 47A58, 41A25, 41A35, 47G10.

1. INTRODUCTION

The efficient extraction of information from limited data has long been a central theme in mathematics and signal analysis. A classical illustrative example is provided by a well-known riddle from the old Turkish tradition. Let us start by telling a very beautiful, funny and clever story from the old times, that is, the time of the sultans.

A sultan commands each of his ten goldsmiths to forge ten golden balls, each expected to weigh exactly one kilogram. However, it is revealed that one of the goldsmiths has used inferior material, producing balls that are lighter by exactly 0.1 kilograms each. The challenge is to determine, using only a single measurement on a precision scale, which goldsmith is dishonest.

The solution proceeds by a weighted sampling strategy: one ball is taken from the first goldsmith, two from the second, and so on, until ten are taken from the tenth. If all the balls were genuine, the total mass should be exactly 55 kilograms. Any deviation from this expected value identifies the fraudulent goldsmith, since the deficit in weight is proportional to the number of balls sampled from the guilty party. For example, an observed deficit of 0.3 kilograms indicates the third goldsmith as the source of the anomaly. Thus, with a single carefully constructed measurement, maximal information is obtained.

This problem serves as a compelling analogy for wavelet theory. Much like the sultan's strategy, wavelet analysis is designed to uncover hidden structure in signals through efficient representation. A wavelet transform decomposes a signal into a hierarchy of subspaces at multiple resolutions, enabling both localization and sparsity in representation. This multiresolution framework allows significant features–such as singularities, discontinuities, or oscillatory

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components—to be identified using relatively few coefficients, in contrast to traditional Fourier methods that distribute information across a global basis.

Just as we distinguish the fake goldsmith by a clever sampling strategy, wavelets distinguish hidden patterns in signals by analyzing them at multiple resolutions using very few coefficients.

Sultan's Problem	**Wavelet Analysis**
One measurement identifies anomaly	*One transform reveals key features*
Strategy uses weighted sample	*Filter banks and scaling functions*
Find anomaly	*Detect feature*
Efficient logic	*Sparse encoding via MRA*

The underlying principle common to both is that of information economy: the ability to reveal essential structure with minimal data. In modern applications, this philosophy is embodied in wavelet methods for data compression, denoising, and numerical analysis of differential equations (see Daubechies 1992 [12]; Mallat 1999 [27]). Just as the sultan's single weighing uncovers the truth with optimal efficiency, a properly designed wavelet transform extracts the most relevant signal characteristics while discarding redundancies. A single, well-designed measurement is often sufficient to reveal underlying structures that are otherwise hidden. This principle holds true across contexts, whether in the allegorical setting of a sultan's court or in the modern analysis of digital signals. Mathematics, through its capacity for abstraction and generalization, consistently serves as a tool for uncovering fundamental truths across different eras and cultures.

The theory of wavelet-based operators plays a fundamental role in signal analysis and has significant applications in digital image processing. In classical Fourier analysis, signals are examined in the frequency domain by transforming them from the time domain. However, this transformation often results in the loss of time-related information, making the analysis of non-stationary signals particularly challenging. To address this limitation, various methods have been developed one of the most prominent being wavelet analysis, originally introduced by Alfred Haar. Wavelets provide a powerful tool for decomposing signals into components that are localized in both time and frequency, allowing for more precise analysis of complex signals. This talk will focus especially on recent applications of wavelet methods to two-dimensional functions and image reconstruction. We will also explore how operators defined using specific types of wavelets serve as natural extensions of classical operators and their Kantorovich-type modifications. Notably, the wavelet-based operators and the associated algorithms presented here can be seamlessly applied to real-world problems traditionally handled by Kantorovich operators, thereby enhancing both the theoretical framework and practical effectiveness of signal and image processing techniques.

Wavelet expansion, or reconstruction of signals via wavelets, allows for more accurate local identification and separation of signal features. A wavelet expansion coefficient represents a component that is itself local and easier to interpret. Wavelets can allow overlapping components of a signal to be separated in both time and frequency. Some detailed informations and advantages of the wavelets can be found in [8], [11] and [12].

In addition, we will see that the results obtained for operators defined using some special cases of wavelets represent a natural extension to the classical Bernstein operators and their Kantorovich-type modifications. It is also worth noting that the operators discussed here are closely related to hybrid type operators and quasi interpolation operators (see [3], [4] and [24]). Please also see the very recent studies of the author's on wavelet type Bezier operators, due to the advantage of the wavelet functions, which give some extensions of the previous results in the literature ([15] and [16]).

Based on the idea developed in [2], [13] and [14]-[22], the central issue of this paper is to extend the theory of interpolation to functionals and operators by introducing the two dimensional Bernstein operators $WB_{n-1,m-1}$ by using the compactly supported Daubechies wavelets. Afterwards, we investigate the convergence problem for these operators.

It is well-known that, for a function defined on the interval $[0,1]$, the Bernstein operators $(B_n f)$, $n \geq 1$, are defined by

$$(B_n f)(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) p_{n,k}(x) \quad , \quad n \geq 1, \tag{1.1}$$

where $p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$ is the well-known Binomial distribution and called Bernstein basis $(0 \leq x \leq 1)$ [5]. As usual, let $B[a,b]$ and $C[a,b]$ be function spaces of real valued bounded and continuous functions defined on $[a,b]$ endowed their usual norms, respectively.

We will denote by $L^\infty(\mathbb{R})$ the space of all the essentially bounded functions $f : \mathbb{R} \to \mathbb{R}$ endowed with the usual ess-sup-norm. Let also $L^p[a,b](1 \leq p < \infty)$ be the space of Lebesgue measurable functions defined on $[a,b]$ with the usual $p-$norm.

To obtain some positive approximation results for functions in $L^p[0,1](1 \leq p < \infty)$, Kantorovich and Durrmeyer type versions of the classical Bernstein operators (1.1) were considered. For detailed approaches to this operator see the fundamental book of G.G. Lorentz [26].

Very recently, as an extension and generalization of the classical Bernstein operators, Karsli introduced in [14] the following wavelet based Bernstein operators $WB_n : B[0,1] \to C[0,1]$, $f \to WB_n f$, which are given by

$$\begin{aligned} (WB_n f)(t) &:= n \sum_{k=0}^{n} p_{n,k}(t) \int_0^1 f(x)\, w(nx-k) dx \\ &= \sum_{k=0}^{n} p_{n,k}(t) \int_0^\lambda f\left(\frac{x+k}{n}\right) w(x) dx, \end{aligned} \tag{1.2}$$

with $t \in [0,1]$, specifying that $\mathrm{supp}(w) \subseteq [0,\lambda]$, $0 < \lambda \leq 1$.

The author proved in [14] that the sequence $(WB_n f)$ converges pointwise and uniformly to f on $[0,1]$, and estimated the rate of these convergence results using the modulus of continuity, second order modulus of smoothness and Peetre's K-functionals. The author also obtained some convergence results in L^p spaces via these operators.

In his Ph.D. thesis [6] written under the direction of G.G. Lorentz and afterwards in the paper [7], the famous German mathematician P.L. Butzer considered two dimensional Bernstein polynomials on the square $\square := \{(x,y) : 0 \leq x, y \leq 1\}$ as follows:

$$B_{n,m}(f;x,y) = \sum_{k=0}^{n} \sum_{j=0}^{m} f\left(\frac{k}{n}, \frac{j}{m}\right) p_{n,k}(x)\, p_{m,j}(y),$$

where $p_{n,k}(t) = \binom{n}{k} t^k (1-t)^{n-k}$.

In the present study, we will reconstruct bivariate Bernstein operators, where location and time are very important and effective, with the help of wavelet expansions. Moreover, we will examine and analyse various properties of the wavelet based extension of the operators. Afterwards, we will provide some examples both on the convergence of functions of two variables and on image processing applications.

2. Preliminaries and auxiliary results

Let us consider two orthogonal functions: the scaling function (or father wavelet) $\phi(t)$ and the wavelet function (or mother wavelet) $\psi(t)$. By scaling and translation of these two orthogonal functions we obtain a complete basis set. These functions have the following important properties;

$$\int_{-\infty}^{\infty} \phi(t)dt = 1, \int_{-\infty}^{\infty} \psi(t)dt = 0,$$

$\phi, \psi \in L^2(\mathbb{R})$, and orthogonal (see [11, 12]). In general, the wavelets refer to the set of family of orthonormal functions of the form

$$\psi_{a,b}(t) = \frac{1}{\sqrt{a}}\psi\left(\frac{t-b}{a}\right), \quad a > 0, b \in \mathbb{R}, \tag{2.3}$$

where ψ is the basic (mother) wavelet. The simplest wavelet is known as the Haar wavelet given by:

$$\psi(x) = \begin{cases} 1 & , \quad 0 \leq x < \frac{1}{2} \\ -1 & , \quad \frac{1}{2} \leq x < 1 \\ 0 & , \quad e.w. \end{cases}$$

with the corresponding scaling function (father wavelet)

$$\phi(t) = \begin{cases} 1 & , \quad 0 \leq x < 1 \\ 0 & , \quad e.w. \end{cases} .$$

Haar wavelets constitutes an orthonormal system for the space of square-integrable functions on the real line.

We now consider a special orthonormal bases, called wavelets. There is a scaling function (father wavelet) $\phi(t)$ with $\{\phi(t-n)\}$ are orthogonal and the mother wavelet $\psi(t)$ based on the father wavelet $\phi(t)$ gives rise to the orthonormal basis

$$\psi_{j,k}(t) := 2^{j/2}\psi(2^j t - k) \tag{2.4}$$

of $L^2(\mathbb{R})$. Moreover, a multiresolution analysis (MRA) is a sequence $(Vj)_{j\in Z}$ of closed subspaces of $L^2(\mathbb{R})$, whose elements are scaling functions (father wavelets).

It is well-known that, each $f \in L^2(\mathbb{R})$ has the following representation

$$f(x) = \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} b_{j,k}\psi_{j,k}(x),$$

called wavelet expansion and $b_{j,k}$ are wavelet coefficients given by

$$b_{j,k} = \langle f(x), \psi_{j,k}(x)\rangle = 2^{j/2}\int_{\mathbb{R}} f(x)\overline{\psi(2^j x - k)}dx$$

(see [2, 13, 23, 25] and [28]).

Let us assume that father wavelets $w \in L_\infty(\mathbb{R})$ satisfies:

w_1) There is a real constant $0 < \lambda \leq 1$ such that supp $w \subset [0, \lambda]$,

w_2)

$$\int_{\mathbb{R}} w(x)dx = 1,$$

w$_3$) The first N moments satisfy

$$m_j^w(w) := \int_{\mathbb{R}} x^j w(x)dx = 0, \quad j = 1, ..., N.$$

Obviously, the absolute moments of the father wavelet w

$$M_j^w(w) := \int_{\mathbb{R}} |x|^j |w(x)| dx < +\infty$$

for every $j \in \mathbb{N}_0 := \{0\} \cup \mathbb{N}$.

Wavelets that meet the above conditions are called compactly supported Daubechies wavelets. Daubechies wavelets have strong relations with the properties of continuity and differentiability. They are supported with $[0, 2N-1]$, in addition there exists a constant $r > 0$ such that for $N \geq 2$, $w \in C^{rN}(\mathbb{R})$ and have a given number of vanishing moments. When $N = 1$, then the first Daubechies wavelet ψ will be the classical Haar basis. As N increases, the regularity of the wavelet increases (see [11, 12]). This means that if we want to use Daubechies wavelets to reconstruct a function, it is more convenient to choose or construct wavelets based on the continuity or differentiability properties of the given function.

Owing to the above definitions, first of all we introduce the bivariate case of the wavelet type Bernstein operators WB_n. We denote

$$\widetilde{f}(z,y) := \begin{cases} f(z,y), & (z,y) \in [0,1]^2 \\ 0, & e.w. \end{cases} . \tag{2.5}$$

Let $B\left([0,1]^2\right) = \left\{ f : [0,1]^2 \to \mathbb{R} \,\middle|\, f \text{ be bounded on } [0,1]^2 \right\}$. The norm on $B\left([0,1]^2\right)$ is given by

$$\|f\|_2 := \sup_{(x,t)\in[0,1]^2} |f(x,t)| .$$

Definition 2.1. *Let $f \in B\left([0,1]^2\right)$, and let $w \in L_\infty(\mathbb{R})$ be a father wavelet satisfying ($\boldsymbol{w}_1$), ($\boldsymbol{w}_2$) and ($\boldsymbol{w}_3$). Then the bivariate wavelet type Bernstein operators are defined by:*

$$\begin{aligned}(WB_{n-1,m-1}f)(x,t) &= nm \sum_{k=0}^{n-1} \sum_{j=0}^{m-1} p_{n-1,k}(t) p_{m-1,j}(x) \int_0^1 \int_0^1 f(z,y)\, w(mz-j) w(ny-k) dz dy \\ &= nm \sum_{k=0}^{n-1} \sum_{j=0}^{m-1} p_{n-1,k}(t) p_{m-1,j}(x) \int_{\mathbb{R}} \int_{\mathbb{R}} \widetilde{f}(z,y)\, w(mz-j) w(ny-k) dz dy, \end{aligned} \tag{2.6}$$

where the function $\widetilde{f}$ being defined in (2.5), with $t, x \in [0,1]$, specifying that $supp(w) \subseteq [0,\lambda]$, $0 < \lambda \leq 1$.

Remark 2.1. *If we choose the father wavelet w as the Haar scaling function, namely $w(\cdot) = \chi_{[0,1]}(\cdot)$, then clearly our wavelet type operators reduce to the Kantorovich form of the Bernstein operators. Indeed:*

$$\begin{aligned}(WB_{n-1,m-1}f)(x,t) &= nm \sum_{k=0}^{n-1} \sum_{j=0}^{m-1} p_{n-1,k}(t) p_{m-1,j}(x) \int_0^1 \int_0^1 f(z,y)\, w(mz-j) w(ny-k) dz dy \\ &= nm \sum_{k=0}^{n-1} \sum_{j=0}^{m-1} p_{n-1,k}(t) p_{m-1,j}(x) \int_{\frac{k}{n}}^{\frac{k+1}{n}} \int_{\frac{j}{m}}^{\frac{j+1}{m}} f(z,y)\, dz dy = (K_{n,m}f)(x,t). \end{aligned}$$

This shows that the bivariate wavelet Bernstein operators (2.6) are a natural extension of the bivariate Bernstein-Kantorovich operators.

3. Fundamental properties

The following approximate results for the wavelet type Bernstein operators need to be remembered for dealing with application and reconstruction of functions. In particular, the following convergence theorem applies when continuous signals (functions) are considered.

Throughout, we denote the monomials by

$$e_{s,l} := e_{s,l}(z,y) = \begin{cases} z^s y^l, & (z,y) \in [0,1]^2 \\ 0 & , \ e.w. \end{cases} \tag{3.7}$$

for $s, l = 0, 1, 2, \ldots$. We have the followings.

Theorem 3.1. *Let $w \in L_\infty(\mathbb{R})$ be a father wavelet satisfies $\boldsymbol{w}_1$, $\boldsymbol{w}_2$ and $\boldsymbol{w}_3$. Then the moments of bivariate wavelet Bernstein operators, constructed by using the compactly supported Daubechies wavelets (2.6) and the Bernstein operators (1.1) are the same, namely*

$$(WB_{n-1,m-1}e_{s,0})(x,t) = (B_{m-1}e_{s,0})(x)$$
$$(WB_{n-1,m-1}e_{0,l})(x,t) = (B_{n-1}e_{0,l})(t)$$
$$(WB_{n-1,m-1}\left(e_{0,l} + e_{s,0}\right))(x,t) = (B_{n-1}e_{0,l})(t) + (B_{m-1}e_{s,0})(x),$$

hold true.

Proof. In view of the definition of the operator (2.6), we have

$$\begin{aligned}
&(WB_{n-1,m-1}e_{s,0})(x,t) \\
=&nm \sum_{k=0}^{n-1} \sum_{j=0}^{m-1} p_{n-1,k}(t) p_{m-1,j}(x) \int_0^1 \int_0^1 z^s w(mz-j) w(ny-k) dz dy \\
=&n \sum_{k=0}^{n-1} \sum_{j=0}^{m-1} p_{n-1,k}(t) p_{m-1,j}(x) \int_{\mathbb{R}} \int_{\mathbb{R}} \left(\frac{u+j}{m}\right)^s w(u) w(ny-k) du dy \\
=&\frac{n}{m^s} \sum_{k=0}^{n-1} \sum_{j=0}^{m-1} p_{n-1,k}(t) p_{m-1,j}(x) \int_{\mathbb{R}} \int_{\mathbb{R}} (u+j)^s \, w(u) w(ny-k) du dy \\
=&\frac{n}{m^s} \sum_{k=0}^{n-1} \sum_{j=0}^{m-1} p_{n-1,k}(t) p_{m-1,j}(x) \int_{\mathbb{R}} \int_{\mathbb{R}} \left[\sum_{i=0}^{s} \binom{s}{i} u^i j^{s-i}\right] w(u) w(ny-k) du dy.
\end{aligned}$$

In view of $\mathbf{w}_3$, one has for $i \neq 0$

$$\int_{\mathbb{R}} \left[\sum_{i=0}^{s} \binom{s}{i} u^i j^{s-i}\right] w(u) du = 0$$

and for $i=0$ and from $\mathbf{w}_2$ we get

$$\begin{aligned}(WB_{n-1,m-1}e_{s,0})(x,t) &= \frac{n}{m^s}\sum_{k=0}^{n-1}\sum_{j=0}^{m-1} p_{n-1,k}(t)p_{m-1,j}(x)\int_{\mathbb{R}}\int_{\mathbb{R}} j^s w(u)w(ny-k)dudy \\ &= \frac{n}{m^s}\sum_{k=0}^{n-1}\sum_{j=0}^{m-1} p_{n-1,k}(t)p_{m-1,j}(x)\int_{\mathbb{R}}\int_{\mathbb{R}} j^s w(u)w(ny-k)dudy \\ &= \sum_{j=0}^{m-1}\frac{j^s}{m^s}p_{m-1,j}(x) \\ &= (B_{m-1}e_{s,0})(x) = x^s.\end{aligned}$$

The proof of the remain parts are similar, so we omit. □

Remark 3.2. *Moreover, the central moments of the bivariate wavelet type Bernstein operators (2.6) are the same as of the classical Bernstein operators (1.2). Indeed, as in the previous Theorem 3.1, we get*

$$\begin{aligned}&(WB_{n-1,m-1}\,(e_{1,0}-x)^{\beta})(x,t) \\ =&nm\sum_{k=0}^{n-1}\sum_{j=0}^{m-1} p_{n-1,k}(t)p_{m-1,j}(x)\int_0^1\int_0^1 (z-x)^{\beta}\, w(mz-j)w(ny-k)dzdy \\ =&m\sum_{j=0}^{m-1} p_{m-1,j}(x)\int_{\mathbb{R}} (z-x)^{\beta}\, w(mz-j)dz \\ =&\sum_{j=0}^{m-1} p_{m-1,j}(x)\int_{\mathbb{R}} \left(\frac{u+j}{m}-x\right)^{\beta} w(u)du \\ =&\frac{1}{m^{\beta}}\sum_{j=0}^{m-1} p_{m-1,j}(x)\int_{\mathbb{R}} (u+j-mx)^{\beta}\, w(u)du \\ =&\frac{1}{m^{\beta}}\sum_{j=0}^{m-1} p_{m-1,j}(x)\int_{\mathbb{R}} \left[\sum_{i=0}^{\beta}\binom{\beta}{i} u^i\,(j-mx)^{\beta-i}\right] w(u)du.\end{aligned}$$

Again by the properties of the compactly supported Daubechies wavelets, namely $\mathbf{w}_2$ and $\mathbf{w}_3$, we get

$$\begin{aligned}(WB_{n-1,m-1}\,(e_{1,0}-x)^{\beta})(x,t) &= \frac{1}{m^{\beta}}\sum_{j=0}^{m-1} p_{m-1,j}(x)\,(j-mx)^{\beta} \\ &= (B_{m-1}\,(e_{1,0}-x)^{\beta})(x).\end{aligned}$$

Similarly one has

$$\begin{aligned}(WB_{n-1,m-1}\,(e_{0,1}-t)^{l})(x,t) &= \frac{1}{n^{l}}\sum_{k=0}^{n-1} p_{n-1,k}(x)\,(k-nt)^{l} \\ &= (B_{n-1}\,(e_{0,1}-t)^{l})(t).\end{aligned}$$

Throughout this work, as in the case of the Bernstein operators, we assume that, the first two central moments of the Bernstein operators, constructed by using the compactly supported

Daubechies wavelets (2.6) satisfy

$$\begin{aligned}
m_0^{l=0} &:= (WB_{n-1,m-1}\,(e_{0,1}-t)^0)(x,t) = 1,\\
m_0^{s=0} &:= (WB_{n-1,m-1}\,(e_{1,0}-x)^0)(x,t) = 1,\\
m_1^{l=1} &:= (WB_{n-1,m-1}\,(e_{0,1}-t)^1)(x,t) = 0,\\
m_1^{s=1} &:= (WB_{n-1,m-1}\,(e_{1,0}-x)^1)(x,t) = 0,\\
m_1^{l=2} &:= (WB_{n-1,m-1}\,(e_{0,1}-t)^2)(x,t) = \frac{t(1-t)}{n}\\
m_1^{s=2} &:= (WB_{n-1,m-1}\,(e_{1,0}-x)^2)(x,t) = \frac{x(1-x)}{m}
\end{aligned} \tag{3.8}$$

and

$$\begin{aligned}
m_1^{l=2} &= \frac{t(1-t)}{n} \le \frac{1}{4n}\\
m_1^{s=2} &= \frac{x(1-x)}{m} \le \frac{1}{4m}
\end{aligned} \tag{3.9}$$

for every $t, x \in [0,1]$.

It is also well-known that for each $i \in \mathbb{N}_0$ there is a constant A_i only depending upon s or l such that

$$\begin{aligned}
0 \le m_1^{l=2i} \le \frac{A_i}{n^i} < \infty,\\
0 \le m_1^{s=2i} \le \frac{A_i}{m^i} < \infty
\end{aligned}$$

hold (page 15 eq (6) Lorentz [26], see also [1]).

Moreover, for every $t \in [0,1]$ and for some $\beta > 0$, the discrete absolute moments of order β satisfy

$$(B_n\,|x-t|^{\beta})(t) \le 2\Gamma\left(\frac{\beta}{2}+1\right)\frac{1}{n^{\beta/2}} < \infty, \tag{3.10}$$

where $\Gamma(\bullet)$ stands for the Gamma function (see [1]).

4. Convergence properties

We now introduce some notations and structural hypotheses, which will be fundamental in proving our convergence theorems. This section also provides the main approximation results of the paper.

To address this task, it is necessary to recall the notions of the modulus of continuities of a given bivariate functions.

Definition 4.2. *Let $f \in C\left([a,b]^2\right)$ and $\delta > 0$ be given. Then the complete modulus of continuity is given by;*

$$\omega(\delta) = \sup_{\sqrt{(u_1-u_2)^2+(v_1-v_2)^2} \le \delta} |f(u_1,v_1) - f(u_2,v_2)|.$$

Further on, the first and second partial modulus of continuity are given by

$$\omega_1(\delta_1, 0) = \sup_{|u_1-u_2|\le\delta_1} |f(u_1,v_1) - f(u_2,v_1)|,$$

$$\omega_2(0, \eta) = \sup_{|v_1-v_2|\le\eta} |f(u_1,v_1) - f(u_1,v_2)|.$$

Definition 2. Recall that the function $\omega(f;\delta)$ has the following well-known properties;

(i) Let $\lambda \in \mathbb{R}^+$, then $\omega(f;\lambda\delta) \leq (\lambda+1)\,\omega(f;\delta)$,

(ii) $\lim\limits_{\delta\to 0^+} \omega(f;\delta) = 0$,

(iii) $|f(u_1,v_1) - f(u_2,v_2)| \leq \omega(\delta)\left(1 + \frac{\sqrt{(u_1-u_2)^2+(v_1-v_2)^2}}{\delta}\right)$.

Note that the same properties also hold for partial moduli of continuity.

We are now ready to establish one of the first main results of this study, which gives a strong relation between Bernstein operators (1.2) and our new operators (2.6) constructed via wavelets. We have the following result.

Theorem 4.2. *Let $f \in B([0,1]^2)$ be a measurable function and let $\psi \in L_\infty(R)$ be a father wavelet satisfies $\mathbf{w}_1$, $\mathbf{w}_2$ and $\mathbf{w}_3$. Then*

$$\lim_{(n,m)\to\infty} (WB_{n-1,m-1}f)(x_0,t_0) = f(x_0,t_0)$$

holds true at each point (x_0,t_0) of continuity of f.

Proof. In view of the definition of the operator (2.6), one has

$$\begin{aligned}&(WB_{n-1,m-1}f)(x_0,t_0) - f(x_0,t_0)\\ =& nm\sum_{k=0}^{n-1}\sum_{j=0}^{m-1} p_{n-1,k}(t_0)p_{m-1,j}(x_0)\int_0^1\int_0^1 f(z,y)\,w(mz-j)w(ny-k)dzdy - f(x_0,t_0).\end{aligned}$$

By Theorem 3.1, we know that

$$(WB_{n-1,m-1}1)(x,t) = 1,$$

and hence, we can write

$$\begin{aligned}&|(WB_{n-1,m-1}f)(x_0,t_0) - f(x_0,t_0)|\\ =&\left|nm\sum_{k=0}^{n-1}\sum_{j=0}^{m-1} p_{n-1,k}(t_0)p_{m-1,j}(x_0)\int_0^1\int_0^1 [f(z,y) - f(x_0,t_0)]\,w(mz-j)w(ny-k)dzdy\right|\\ \leq&\sum_{k=0}^{n-1}\sum_{j=0}^{m-1} p_{n-1,k}(t_0)p_{m-1,j}(x_0)\int_{\mathbb{R}}\int_{\mathbb{R}}\left|\tilde{f}\left(\frac{u+j}{m},\frac{v+k}{n}\right) - f(x_0,t_0)\right| w(u)w(v)dudv.\end{aligned}$$

Let us divide the last term into two parts as;

$$|(WB_{n-1,m-1}f)(x_0,t_0) - f(x_0,t_0)| \leq P_1 + P_2,$$

where

$$\begin{aligned}P_1 =& \sum_{k=0}^{n-1}\sum_{j=0}^{m-1} p_{n-1,k}(t_0)p_{m-1,j}(x_0)\\ &\times \iint\limits_{\sqrt{\left(\frac{u+j}{m}-x_0\right)^2+\left(\frac{v+k}{n}-t_0\right)^2}<\delta} \left|\tilde{f}\left(\frac{u+j}{m},\frac{v+k}{n}\right) - f(x_0,t_0)\right| w(u)w(v)dudv\end{aligned}$$

and

$$P_2 = \sum_{k=0}^{n-1}\sum_{j=0}^{m-1} p_{n-1,k}(t_0)p_{m-1,j}(x_0)$$
$$\times \iint\limits_{\sqrt{\left(\frac{u+j}{m}-x_0\right)^2+\left(\frac{v+k}{n}-t_0\right)^2}\geq\delta} \left|\widetilde{f}\left(\frac{u+j}{m},\frac{v+k}{n}\right)-f(x_0,t_0)\right| w(u)w(v)dudv$$

Since (x_0,t_0) is a continuity point of f, then clearly

$$|f(z,y)-f(x_0,t_0)|<\epsilon$$

whenever $\sqrt{(z-x_0)^2+(y-t_0)^2}<\delta$, hence we can write

$$P_1<\epsilon.$$

On the other hand, since

$$|f(z,y)-f(x_0,t_0)|\leq 2\|f\|_2$$

whenever $\sqrt{(z-x_0)^2+(y-t_0)^2}\geq\delta$, we get

$$P_2 = \sum_{k=0}^{n-1}\sum_{j=0}^{m-1} p_{n-1,k}(t_0)p_{m-1,j}(x_0)$$
$$\times \iint\limits_{\sqrt{\left(\frac{u+j}{m}-x_0\right)^2+\left(\frac{v+k}{n}-t_0\right)^2}\geq\delta} \left|\widetilde{f}\left(\frac{u+j}{m},\frac{v+k}{n}\right)-f(x_0,t_0)\right| w(u)w(v)dudv.$$
$$\leq 2\|f\|_2 \sum_{k=0}^{n-1}\sum_{j=0}^{m-1} p_{n-1,k}(t_0)p_{m-1,j}(x_0) \iint\limits_{\sqrt{\left(\frac{u+j}{m}-x_0\right)^2+\left(\frac{v+k}{n}-t_0\right)^2}\geq\delta} w(u)w(v)dudv$$
$$\leq 2\|f\|_2 \frac{m_1^{l=2}+m_1^{s=2}}{\delta^2} \leq \frac{\|f\|}{\delta^2}\left(\frac{1}{2n}+\frac{1}{2m}\right).$$

Collecting these estimates, we have

$$\lim_{(n,m)\to\infty}(WB_{n-1,m-1}f)(x_0,t_0)=f(x_0,t_0).$$

This completes the proof. □

As a consequence of the Theorem 3.1 we have also the following uniform convergence result.

Corollary 4.1. *The same arguments apply to the case when $f\in C\left([0,1]^2\right)$. In this case the convergence is uniform with respect to $x,t\in[0,1]$, and hence one has*

$$\lim_{(n,m)\to\infty}\|(WB_{n-1,m-1}f)-f\|_{C([0,1]^2)}=0.$$

5. Practical examples, graphical representations

5.1. **Application to functions.** Now, we will give some graphical examples for these approach, namely convergence to functions by means of wavelet based bivariate Bernstein operators $(WB_{n-1,m-1}f)(x,y)$.

Example 5.1. *Let* $f(x,y) = 3(\sin(10xy)+1)$. *We consider a special case of the wavelet based Bernstein operators* $(WB_{n-1,m-1}f)(x,y)$ *constructed by using Shannon wavelet function.Then one has for* $n,m = 2,5,10,20$ *and for* 40.

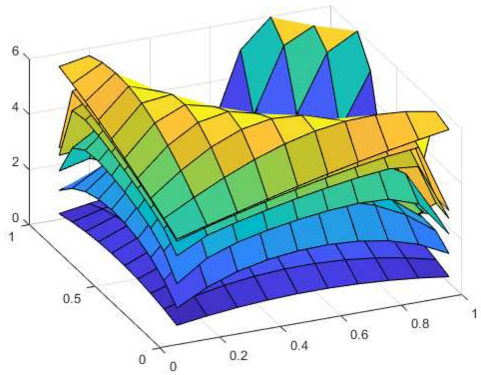

FIGURE 1. Approximation to $f(x,y) = 3(\sin(10xy)+1)$ by Shannon wavelet based Bernstein operator, for $n,m = 2,5,10,20$ and 40.

Example 5.2. *Let* $f(x,y) = 3(\sin(10xy)+1)$. *We consider the wavelet based Bernstein operators operators* $(WB_{n-1,m-1}f)(x,y)$ *constructed by using Haar scaling function. Then one has for* $n,m = 20,25$ *and for* 30.

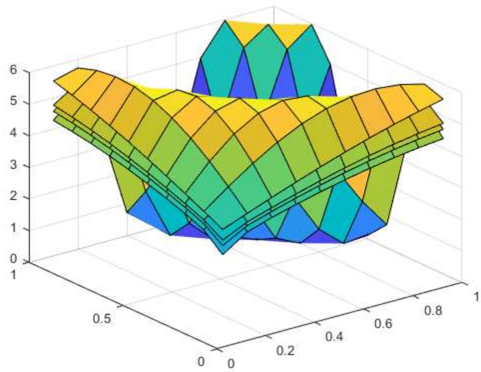

FIGURE 2. Approximation of $f(x,y) = 3(\sin(10xy)+1)$ by Haar wavelet based Bernstein operator, for $n,m = 20,25$ and 30.

Example 5.3. *Let* $f(x,y) = 3(\sin(10xy)+1)$, *and we consider the wavelet based Sampling operators* $(WS_{n,m}f)(x,y)$ *constructed by using Shannon wavelet function. Then one has for* $n,m = 10,30$ *and for* 50.

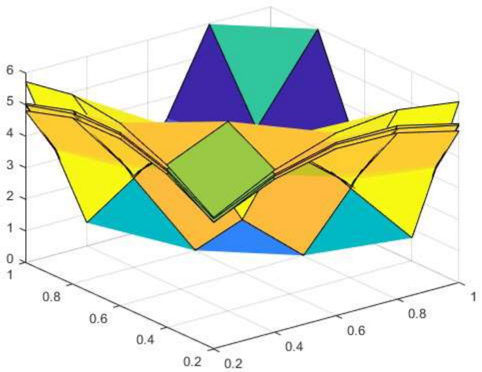

FIGURE 3. Approximation to $f(x,y) = 3(\sin(10xy) + 1)$ by Shannon wavelet based Sampling operator, for $n, m = 10, 30$ and 50.

Example 5.4. *Let $f(x,y) = x^2 - x + 1$. We consider a special case of the wavelet based Bernstein operators $(WB_{n-1,m-1}f)(x,y)$ constructed by using Shannon wavelet function.Then one has for $n, m = 2, 5, 10, 20$ and for 40.*

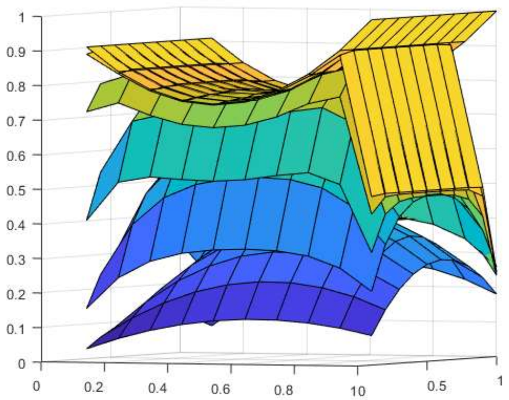

FIGURE 4. Approximation to $f(x,y) = x^2 - x + 1$ by Shannon wavelet based Bernstein operator, for $n, m = 2, 5, 10, 20$ and 40.

Example 5.5. *Let $f(x,y) = x^2 - x + 1$, and we consider the wavelet based Bernstein operators $(WB_{n-1,m-1}f)(x,y)$ constructed by using Haar scaling function. Then one has for $n, m = 2, 5, 10, 20$ and for 40.*

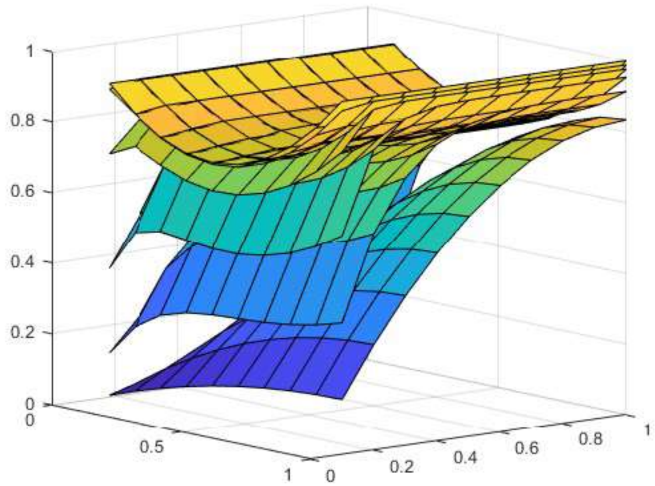

FIGURE 5. Approximation to $f(x,y) = x^2 - x + 1$ by Haar wavelet based Bernstein operator, for $n = 2, 5, 10, 20$ and 40.

5.2. **Application on images.** As modern technology has advanced, enlarging digital images has become widespread in various fields including digital photography, medical imaging, and smartphones.

Numerous zooming techniques have been explored in the literature, such as pixel duplication, interpolation methods, zero-order hold, and others.

Now, we will give some image reconstruction examples via wavelet based operators.

Starting from the 64×64 pixel grayscale Baboon image:

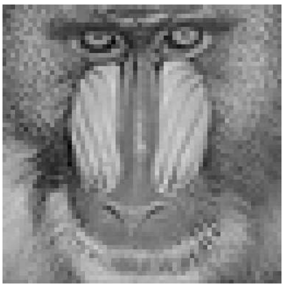

FIGURE 6. Original Baboon Image (64×64) pixels

The image was first downsampled to 32×32 pixels.

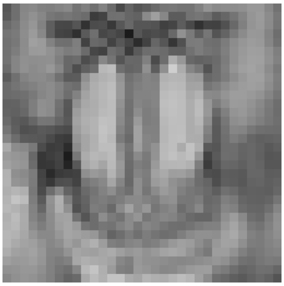

FIGURE 7. Downscaled Baboon Image (32×32) pixels

In order to demonstrate the smoothing capabilities of the aforementioned operators, we will reconstruct the 32×32 pixel downscaled Baboon image back to its original 64×64 resolution using Haar wavelet-based Bernstein and Shannon wavelet-based Bernstein operators.

The quality of the reconstructed images was assessed by calculating the Peak Signal-to-Noise Ratio (PSNR) with respect to the original image.

The aforementioned procedures were implemented using MATLAB programming language.

We consider the wavelet based Bernstein operators $(WB_{n-1,m-1}f)(x,y)$ constructed by using Haar scaling function. Then one has for $n, m = 1$,

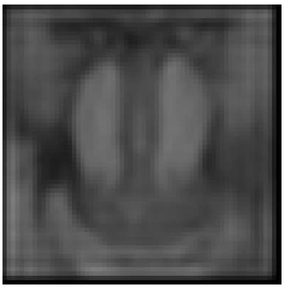

FIGURE 8. Reconstruction Baboon Image (64×64 pixels) via Haar based Bernstein operators for $n, m = 1$.

In this reconstruction the Peak Signal-to-Noise Ratio (**PSNR**) with respect to the original image is **16.0959**.

We consider the wavelet based Bernstein operators $(WB_{n-1,m-1}f)(x,y)$ constructed by using Haar scaling function. Then one has for $n, m = 5$,

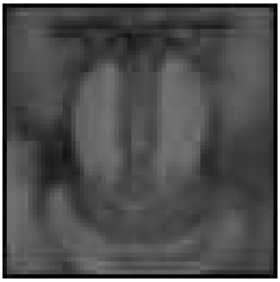

FIGURE 9. Reconstruction Baboon Image ($64x64$ pixels) via Haar based Bernstein operators for $n, m = 5$.

In this reconstruction the Peak Signal-to-Noise Ratio (**PSNR**) with respect to the original image is **19.3205**.

We consider the wavelet based Bernstein operators $\left(WB_{n-1,m-1}f\right)(x, y)$ constructed by using Shannon wavelet function. Then one has for $n, m = 1$,

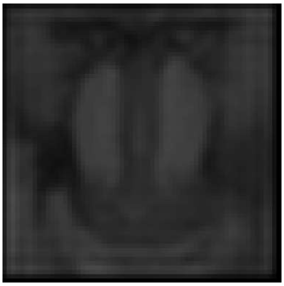

FIGURE 10. Reconstruction Baboon Image (64×64 pixels) via Shannon based Bernstein operators for $n, m = 1$.

In this reconstruction the Peak Signal-to-Noise Ratio (**PSNR**) with respect to the original image is **14.9116**.

Starting from the 64×64 pixel grayscale Umut's image:

FIGURE 11. Original Umut's Image (64×64) pixels

The image was first downsampled to 32×32 pixels.

FIGURE 12. Downscaled Umut's Image (32×32) pixels

We consider the wavelet based Bernstein operators $(WB_{n-1,m-1}f)(x,y)$ constructed by using Haar scaling function. Then one has for $n, m = 1$,

FIGURE 13. Reconstruction Umut's Image (64×64 pixels) via Haar based Bernstein operators for $n, m = 1$.

In this reconstruction the Peak Signal-to-Noise Ratio (**PSNR**) with respect to the original image is **16.1996**.

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Research Article

On new φ-fixed point results involving discontinuous control functions with the effectively example and its applications

PATHAITHEP KUMROD AND WUTIPHOL SINTUNAVARAT*

ABSTRACT. The main purpose of this paper is to extend and enhance the results of Karapinar *et al.* [7] by relaxing the continuity assumption on control functions in the contractive setting. The validity and wider applicability of our principal theorem are illustrated through examples. In addition, our generalized framework yields a homotopy result and establishes the existence of solutions for a class of integral equations.

Keywords: φ-fixed points, φ-Picard mappings, homotopy result.

2020 Mathematics Subject Classification: 47H10, 54H25.

1. INTRODUCTION AND PRELIMINARIES

In the last ten decades, the classical Banach Contraction Principle (shortened, BCP) in [4] has been investigated and improved by many researchers in several different ways, by the following ideas:

- introducing the generalized Banach contractive conditions;
- increasing the number of involved mappings;
- extending the class of ambient spaces;
- enlarging the idea of fixed points.

Nowadays, there are many interesting research works in all of the above directions (see [11, 1, 8, 2, 10] and the references therein). In the following discussion, we focus particularly on those studies that have directly inspired the development of the present work and are closely related to its main ideas.

In 1969, Boyd and Wong [5] proved the fixed point theorem, which is one of the interesting generalizations of the classical Banach contraction principle, and introduced the following family of control functions:

$$\Psi = \left\{ \psi : [0, \infty) \to [0, \infty) \mid \psi(t) < t \text{ for each } t > 0 \text{ and } \limsup_{r \to t^+} \psi(r) < t \text{ for each } t > 0 \right\}.$$

Afterwards, many mathematicians proved various fixed point results with the help of the control functions in Ψ.

In another direction, Jleli et al. [6] extended the Banach contraction principle by introducing new control functions and first proposed the notions of φ-fixed points and φ-Picard mappings. Before presenting their definitions and main results, we recall the following essential notions.

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Let X be a nonempty set, $\varphi : X \to [0,\infty)$ be a given function, and $T : X \to X$ be a mapping. We denote by F_T the set of all fixed points of T, and by Z_φ the set of all zeros of the function φ, that is,

$$Z_\varphi = \{x \in X \mid \varphi(x) = 0\}.$$

Definition 1.1 ([6]). *Let X be a nonempty set and $\varphi : X \to [0,\infty)$ be a given function. An element $z \in X$ is called a φ-fixed point of $T : X \to X$ if and only if z is a fixed point of T and $\varphi(z) = 0$, that is, $z \in F_T \cap Z_\varphi$.*

Definition 1.2 ([6]). *Let (X, d) be a metric space and $\varphi : X \to [0,\infty)$ be a given function. A mapping $T : X \to X$ is called a φ-Picard mapping if and only if the following conditions hold:*

(i) *$F_T \cap Z_\varphi = \{z\}$, where $z \in X$;*
(ii) *$T^n x \to z$ as $n \to \infty$ for each $x \in X$.*

In the sequel, we denote by $\mathcal{F}$ the class of all functions $F : [0,\infty)^3 \to [0,\infty)$ satisfying the following conditions:

(F1) $\max\{a, b\} \leq F(a, b, c)$ for all $a, b, c \in [0,\infty)$;
(F2) $F(0, 0, 0) = 0$;
(F3) F is continuous.

This class of functions was first introduced by Jleli et al. [6] to generalize classical contractive conditions and to unify several fixed point results under a broader framework. To illustrate the definition, we present some typical examples below, followed by Jleli et al. [6].

Example 1.1 ([6]). *The following functions $F_1, F_2, F_3 : [0,\infty)^3 \to [0,\infty)$ belong to $\mathcal{F}$:*

(i) *$F_1(a, b, c) = a + b + c$ for all $a, b, c \in [0,\infty)$;*
(ii) *$F_2(a, b, c) = \max\{a, b\} + c$ for all $a, b, c \in [0,\infty)$;*
(iii) *$F_3(a, b, c) = a + a^2 + b + c$ for all $a, b, c \in [0,\infty)$.*

These examples demonstrate that the class $\mathcal{F}$ encompasses a wide range of control functions, each leading to different types of contractive mappings. Using this framework, Jleli et al. [6] established the following fundamental fixed point theorem.

Theorem 1.1 ([6]). *Let (X, d) be a complete metric space, and let $\varphi : X \to [0,\infty)$ be a given lower semi-continuous function. Suppose that $T : X \to X$ is a mapping satisfying*

$$F(d(Tx, Ty), \varphi(Tx), \varphi(Ty)) \leq k\, F(d(x, y), \varphi(x), \varphi(y)) \tag{1.1}$$

for all $x, y \in X$, where $F \in \mathcal{F}$ and $k \in [0, 1)$. Then T is a φ-Picard mapping and $F_T \subseteq Z_\varphi$.

If we take $F(a, b, c) = a+b+c$ for all $a, b, c \in [0,\infty)$ and $\varphi(x) = 0$ for all $x \in X$ in Theorem 1.1, then (1.1) reduces to the classical Banach contractive condition. Consequently, Theorem 1.1 also reduces to the well-known Banach contraction principle.

Motivated by both the works of Boyd and Wong [5] and Jleli et al. [6], Karapinar et al. [7] proved φ-fixed point theorems by replacing the condition (F2) with the following modified assumption:

(F2*) $F(a, 0, 0) = a$ for all $a \geq 0$,

and by substituting the constant k with a control function $\psi \in \Psi$. This generalization significantly broadens the applicability of the φ-Picard mapping framework. In 2017, Asadi [3] further refined these results by showing that the continuity of $F \in \mathcal{F}$ in Theorem 1.1 can be weakened. In particular, the theorem remains valid when the following weaker requirement replaces the continuity condition:

- $\limsup_{n\to\infty} F(x_n, y_n, 0) \leq F(x, y, 0)$ whenever $x_n \to x$ and $y_n \to y$ as $n \to \infty$.

Example 1.2 ([3]). *Let $F : [0,\infty)^3 \to [0,\infty)$ be defined by $F(a,b,c) = a + b + [c]$ or $F(a,b,c) = \max\{a,b\} + [c]$ for all $a,b,c \in [0,\infty)$, where $[c]$ denotes the integer part of c. Then F satisfies the condition of Asadi* [3], *but F is not continuous.*

Motivated by the results of Asadi [3] and Karapinar et al. [7], the main objective of this work is to introduce a new class of control functions used in a generalized contractive condition and to establish a φ-fixed point theorem for mappings satisfying this condition with respect to the proposed class of control functions. Our results properly extend and generalize various well-known fixed point theorems in the existing literature. Additionally, illustrative examples and applications are provided to demonstrate the validity and practical utility of the obtained results.

2. Main results

First, we denote by $\mathcal{G}$ the set of all functions $G : [0,\infty)^3 \to [0,\infty)$ that satisfy the following conditions:

- (G1) $\max\{a,b\} \le G(a,b,c)$ for all $a,b,c \in [0,\infty)$,
- (G2) $G(a,0,0) = a$ for all $a \ge 0$,
- (G3) $\limsup\limits_{n\to\infty} G(x_n,y_n,z_n) \le G(x,0,0)$ whenever $x_n \to x$, $y_n \to 0$, and $z_n \to 0$ as $n \to \infty$.

As examples, consider the functions $G_1, G_2 : [0,\infty)^3 \to [0,\infty)$ defined by

$$G_1(a,b,c) = a + b + [c] \quad \text{and} \quad G_2(a,b,c) = \max\{a,b\} + [c]$$

for all $a,b,c \in [0,\infty)$, where $[c]$ denotes the integer part of c. Clearly, $G_1, G_2 \in \mathcal{G}$.

Next, we introduce a new class of control functions that will be used to define a generalized contractive condition in our main φ-fixed point theorem, extending the class ψ considered in previous works.

Let Λ denote the set of all functions $\lambda : [0,\infty) \to [0,\infty)$ satisfying:

- (λ1) $\lambda(t) < t$ for all $t > 0$;
- (λ2) if $\{a_n\}$ is a sequence in $[0,\infty)$ with $\limsup\limits_{n\to\infty} a_n \le a$, then $\limsup\limits_{n\to\infty} \lambda(a_n) \le \lambda(a)$.

We are now ready to present our main result.

Theorem 2.2. *Let (X,d) be a complete metric space and T be a self mapping on X such that*

$$G(d(Tx,Ty),\varphi(Tx),\varphi(Ty)) \le \lambda(G(d(x,y),\varphi(x),\varphi(y))) \tag{2.2}$$

for all $x,y \in X$, where $\varphi : X \to [0,\infty)$ is lower semi-continuous, $G \in \mathcal{G}$, and $\lambda \in \Lambda$. Then $F_T \subseteq Z_\varphi$ and T is a φ-Picard mapping.

Proof. First, we need to show that $F_T \subseteq Z_\varphi$. Let $x \in F_T$. By putting $x = y$ in (2.2), we have

$$G(0,\varphi(x),\varphi(x)) \le \lambda(G(0,\varphi(x),\varphi(x))). \tag{2.3}$$

Assume on the contrary that $\varphi(x) > 0$. Then $G(0,\varphi(x),\varphi(x)) > 0$. From (2.3) and ($\lambda$1), we get

$$G(0,\varphi(x),\varphi(x)) \le \lambda(G(0,\varphi(x),\varphi(x))) < G(0,\varphi(x),\varphi(x)),$$

which is a contradiction. Hence,

$$\varphi(x) = 0,$$

which implies that

$$F_T \subseteq Z_\varphi. \tag{2.4}$$

Next, we will show that T is a φ-Picard mapping. Let x_0 be an arbitrary point in X. Define the sequence $\{x_n\} \subseteq X$ by $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$. If $x_{n^*} = x_{n^*-1}$ for some $n^* \in \mathbb{N}$, then x_{n^*-1} is a fixed point of T. We have nothing to prove and so we may assume that

$$d(x_n, x_{n-1}) > 0 \tag{2.5}$$

for all $n \in \mathbb{N}$. It follows from (G1) that

$$G(d(x_n, x_{n-1}), \varphi(x_n), \varphi(x_{n-1})) > 0 \tag{2.6}$$

for all $n \in \mathbb{N}$. Hence by the contractive condition (2.2), (2.6) and (λ1) we have

$$\begin{aligned} G(d(x_{n+1}, x_n), \varphi(x_{n+1}), \varphi(x_n)) &\leq \lambda(G(d(x_n, x_{n-1}), \varphi(x_n), \varphi(x_{n-1}))) \\ &< G(d(x_n, x_{n-1}), \varphi(x_n), \varphi(x_{n-1})) \end{aligned} \tag{2.7}$$

for all $n \in \mathbb{N}$. This shows that $\{G(d(x_{n+1}, x_n), \varphi(x_{n+1}), \varphi(x_n))\}$ is a decreasing sequence and hence it converges to some point $r \geq 0$, that is,

$$\lim_{n\to\infty} G(d(x_{n+1}, x_n), \varphi(x_{n+1}), \varphi(x_n)) = r. \tag{2.8}$$

From (2.7), (2.8) and the squeeze theorem, we get

$$\lim_{n\to\infty} \lambda(G(d(x_n, x_{n-1}), \varphi(x_n), \varphi(x_{n-1}))) = r. \tag{2.9}$$

We will show that $r = 0$. Suppose by way of contradiction that $r > 0$. By (λ1), (λ2), (2.8) and (2.9) we have

$$r = \limsup_{n\to\infty} \lambda(G(d(x_n, x_{n-1}), \varphi(x_n), \varphi(x_{n-1}))) \leq \lambda(r) < r,$$

which provides a contradiction. Therefore, $r = 0$, that is,

$$\lim_{n\to\infty} G(d(x_{n+1}, x_n), \varphi(x_{n+1}), \varphi(x_n)) = \lim_{n\to\infty} \lambda(G(d(x_n, x_{n-1}), \varphi(x_n), \varphi(x_{n-1}))) = 0,$$

and thus, by (G1)

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = \lim_{n\to\infty} \varphi(x_{n+1}) = 0. \tag{2.10}$$

In what follows, we shall prove that $\{x_n\}$ is a Cauchy sequence. Suppose, to the contrary, that $\{x_n\}$ is not a Cauchy sequence. Then there exists $\epsilon > 0$ such that we can find subsequences $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ of $\{x_n\}$ with $n(k) > m(k) \geq k$, for all positive integer k, satisfying

$$d\left(x_{m(k)}, x_{n(k)}\right) \geq \epsilon. \tag{2.11}$$

We may assume that

$$d\left(x_{m(k)}, x_{n(k)-1}\right) < \epsilon, \tag{2.12}$$

by choosing $n(k)$ to be the smallest integer exceeding $m(k)$ for which (2.11) holds. Now, using the triangle inequality and (2.12), we have

$$\epsilon \leq d\left(x_{m(k)}, x_{n(k)}\right) \leq d\left(x_{m(k)}, x_{n(k)-1}\right) + d\left(x_{n(k)-1}, x_{n(k)}\right) < \epsilon + d\left(x_{n(k)-1}, x_{n(k)}\right).$$

Letting $k \to \infty$ in the previous inequality and using (2.10), we get

$$\lim_{k\to\infty} d\left(x_{m(k)}, x_{n(k)}\right) = \epsilon. \tag{2.13}$$

Using $(G2)$, $(G3)$, (2.10) and (2.13), it follows that

$$\limsup_{k\to\infty} G\left(d\left(x_{m(k)}, x_{n(k)}\right), \varphi\left(x_{m(k)}\right), \varphi\left(x_{n(k)}\right)\right) \leq G(\epsilon, 0, 0) = \epsilon. \tag{2.14}$$

From (λ2), it follows from (2.14) that

$$\limsup_{k\to\infty} \lambda\left(G\left(d\left(x_{m(k)}, x_{n(k)}\right), \varphi\left(x_{m(k)}\right), \varphi\left(x_{n(k)}\right)\right)\right) \leq \lambda(\epsilon). \tag{2.15}$$

On the other hand, by the triangle inequality, (2.2) and $(G1)$, for all $k \in \mathbb{N}$, we see that

$$\begin{aligned}
\epsilon &\leq d\left(x_{m(k)}, x_{n(k)}\right) \\
&\leq d\left(x_{m(k)}, x_{m(k)+1}\right) + d\left(x_{m(k)+1}, x_{n(k)+1}\right) + d\left(x_{n(k)+1}, x_{n(k)}\right) \\
&\leq d\left(x_{m(k)}, x_{m(k)+1}\right) + G\left(d\left(x_{m(k)+1}, x_{n(k)+1}\right), \varphi\left(x_{m(k)+1}\right), \varphi\left(x_{n(k)+1}\right)\right) + d\left(x_{n(k)+1}, x_{n(k)}\right) \\
&\leq d\left(x_{m(k)}, x_{m(k)+1}\right) + \lambda\left(G\left(d(x_{m(k)}, x_{n(k)}), \varphi(x_{m(k)}), \varphi(x_{n(k)})\right)\right) + d\left(x_{n(k)+1}, x_{n(k)}\right).
\end{aligned}$$

Taking limit superior as $k \to \infty$ and using (2.10), (2.15) and $(\lambda 2)$, we have

$$\epsilon \leq \lambda(\epsilon) < \epsilon,$$

which is a contradiction. Therefore, $\{x_n\}$ is a Cauchy sequence. By the completeness of X, there exists a point $z \in X$ such that

$$\lim_{n \to \infty} x_n = z. \tag{2.16}$$

Using (2.10), (2.16) and the semi-continuity of φ, we get

$$0 \leq \varphi(z) \leq \liminf_{n \to \infty} \varphi(x_n) = 0,$$

which implies that

$$\varphi(z) = 0. \tag{2.17}$$

Now, we shall prove that z is a fixed point of T. From (2.5) and (2.16), there exists a subsequence $\{f(n)\}$ of $\{n\}$ such that

$$d\left(x_{f(n)}, z\right) > 0 \tag{2.18}$$

holds for any $n \in \mathbb{N}$. By applying (2.2) and using $(G1)$, $(\lambda 1)$, (2.17) and (2.18), we have

$$\begin{aligned}
d\left(x_{f(n)+1}, Tz\right) &\leq \max\left\{d\left(x_{f(n)+1}, Tz\right), \varphi\left(x_{f(n)+1}\right)\right\} \\
&\leq G\left(d\left(x_{f(n)+1}, Tz\right), \varphi\left(x_{f(n)+1}\right), \varphi(Tz)\right) \\
&\leq \lambda\left(G\left(d\left(x_{f(n)}, z\right), \varphi\left(x_{f(n)}\right), \varphi(z)\right)\right) \\
&< G\left(d\left(x_{f(n)}, z\right), \varphi\left(x_{f(n)}\right), \varphi(z)\right) \\
&= G\left(d\left(x_{f(n)}, z\right), \varphi\left(x_{f(n)}\right), 0\right).
\end{aligned}$$

Thus

$$\limsup_{n \to \infty} d\left(x_{f(n)+1}, Tz\right) \leq \limsup_{n \to \infty} G\left(d\left(x_{f(n)}, z\right), \varphi\left(x_{f(n)}\right), 0\right) \leq G(0,0,0) = 0,$$

which implies that

$$\lim_{n \to \infty} d(x_{f(n)+1}, Tz) = 0.$$

By the uniqueness of the limit, we get $z = Tz$.

Finally, we prove that z is the unique fixed point of T. Assume, for the sake of contradiction, that there exists another fixed point $w \in X$ such that $z \neq w$. Then $z = Tz$ and $w = Tw$. Using (2.2), (2.4), $(\lambda 1)$, and $(G2)$, we have

$$d(z,w) = G(d(z,w),0,0) \leq \lambda(G(d(z,w),0,0)) < G(d(z,w),0,0) = d(z,w),$$

which is a contradiction. Hence, z is the unique fixed point of T. From this conclusion, we can conclude that T is a φ-Picard mapping. □

Remark 2.1. *Note that in the proof of Theorem 2.2, we use the fact that $a > 0$ or $b > 0$ implies that $G(a,b,c) > 0$ for all $a,b,c > 0$. So our proof is simpler and shorter than the proof of Karapinar et al.* [7].

Remark 2.2. *By the properties of the classes $\mathcal{G}$ and Λ, we immediately obtain the main theorem of Karapinar et al.* [7].

The following example shows that our theorem is a proper generalization of some results in the literature.

Example 2.3. *Let $X = [0, 3]$ endowed with the usual metric $d : X \times X \to [0, \infty)$ defined by $d(x, y) = |x - y|$ for all $x, y \in X$. Then (X, d) is a complete metric space. Define the self-mapping $T : X \to X$ by*

$$Tx = \begin{cases} \frac{x^2}{9}, & 0 \leq x < 2, \\ \frac{x}{x+2}, & 2 \leq x \leq 3. \end{cases}$$

Furthermore, we define a function $\lambda : [0, \infty) \to [0, \infty)$ by

$$\lambda(t) = \begin{cases} \frac{t}{3}, & 0 \leq t < 2, \\ \sin\left(\frac{2}{t}\right) + \frac{9}{8}, & t \geq 2. \end{cases}$$

The graph of λ shown in blue is given in Figure 1. It is easy to see that $\lambda \in \Lambda \cap \Psi$.

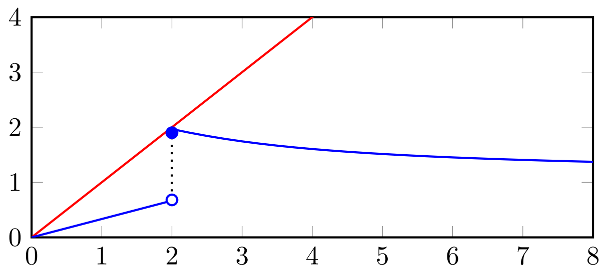

FIGURE 1. The graph of λ

Note that the Banach contraction principle is not applicable because T is not continuous at a point 2. Also, the fixed point theorem of Boyd and Wong [5] *cannot be applied in this case. Indeed, for $x = 1$ and $y = 2$, we get*

$$d(Tx, Ty) = |T1 - T2| = \left|\frac{1}{9} - \frac{1}{2}\right| = \frac{7}{18} \not< \frac{1}{3} = \lambda(1) = \lambda(d(1, 2)) = \lambda(d(x, y)).$$

Now, let us consider the mapping $G : [0, \infty)^3 \to [0, \infty)$ and $\varphi : X \to [0, \infty)$ defined by

$$G(a, b, c) = a + b + [c]$$

for each $a, b, c \geq 0$, where $[c]$ is the integer part of c and $\varphi(x) = \frac{x}{2}$ for each $x \in X$. Clearly, $G \in \mathcal{G}$ and φ is lower semi-continuous. Finally, we shall claim that the mapping T satisfies the contractive condition (2.2). Suppose that $x, y \in X$. We distinguish the following four cases:

Case 1: *If* $(x, y) \in [0, 2) \times [0, 2)$, *then*

$$
\begin{aligned}
& G(d(Tx,Ty),\varphi(Tx),\varphi(Ty)) \\
= \;& d(Tx,Ty) + \varphi(Tx) + [\varphi(Ty)] \\
= \;& \frac{|x^2 - y^2|}{9} + \frac{x^2}{18} \\
\leq \;& \frac{|x-y|}{3} + \frac{x}{6} \\
\leq \;& \lambda\left(|x-y| + \frac{x}{2} + \left[\frac{y}{2}\right]\right) \\
= \;& \lambda(d(x,y) + \varphi(x) + [\varphi(y)]) \\
= \;& \lambda(G(d(x,y),\varphi(x),\varphi(y))).
\end{aligned}
\tag{2.19}
$$

The 3D graphs (plotted in MATLAB) in Figure 2 guarantee that the condition (2.19) holds.

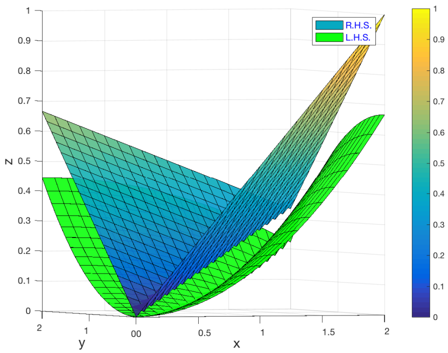

FIGURE 2. The graph of (2.19)

Case 2: *If* $(x, y) \in [2, 3] \times [2, 3]$, *then*

$$
\begin{aligned}
& G(d(Tx,Ty),\varphi(Tx),\varphi(Ty)) \\
= \;& d(Tx,Ty) + \varphi(Tx) + [\varphi(Ty)] \\
= \;& \left|\frac{x}{x+2} - \frac{y}{y+2}\right| + \frac{x}{2(x+2)} \\
\leq \;& \sin\left(\frac{2}{|x-y| + \frac{x}{2} + \left[\frac{y}{2}\right]}\right) + \frac{9}{8} \\
= \;& \lambda\left(|x-y| + \frac{x}{2} + \left[\frac{y}{2}\right]\right) \\
= \;& \lambda(d(x,y) + \varphi(x) + [\varphi(y)]) \\
= \;& \lambda(G(d(x,y),\varphi(x),\varphi(y))).
\end{aligned}
\tag{2.20}
$$

The 3D graphs (plotted in MATLAB) in Figure 3 guarantee that the condition (2.20) holds.

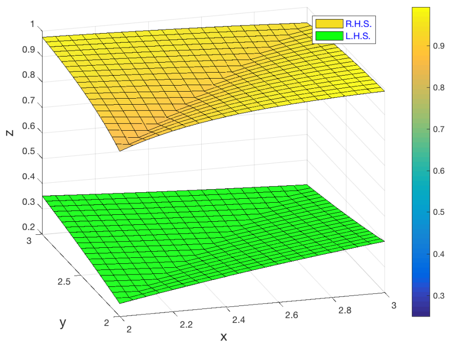

FIGURE 3. The graph of (2.20)

Case 3: *If $x \in [0,2)$ and $y \in [2,3]$, then*

$$
\begin{aligned}
& G(d(Tx,Ty),\varphi(Tx),\varphi(Ty)) \\
= \; & d(Tx,Ty)+\varphi(Tx)+[\varphi(Ty)] \\
= \; & \left|\frac{x^2}{9}-\frac{y}{y+2}\right|+\frac{x^2}{18} \\
\le \; & \frac{|x-y|}{3}+\frac{x}{6}+\frac{1}{3}\left[\frac{y}{2}\right] \\
= \; & \lambda\left(|x-y|+\frac{x}{2}+\left[\frac{y}{2}\right]\right) \\
= \; & \lambda(d(x,y)+\varphi(x)+[\varphi(y)]) \\
= \; & \lambda(G(d(x,y),\varphi(x),\varphi(y))).
\end{aligned}
\tag{2.21}
$$

The 3D graphs (plotted in MATLAB) in Figure 4 guarantee that the condition (2.21) holds.

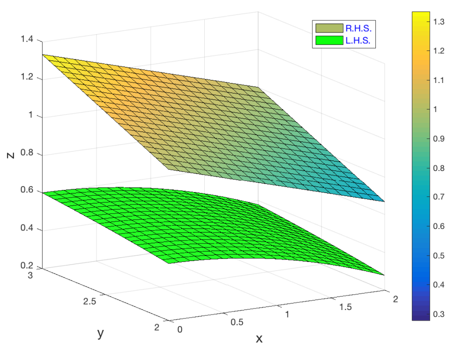

FIGURE 4. The graph of (2.21)

Case 4: *If $x \in [2,3]$ and $y \in [0,2)$, then*

$$\begin{aligned} & G(d(Tx,Ty),\varphi(Tx),\varphi(Ty)) \\ = \; & d(Tx,Ty)+\varphi(Tx)+[\varphi(Ty)] \\ = \; & \left|\frac{x}{x+2}-\frac{y^2}{9}\right|+\frac{x}{2(x+2)} \\ \leq \; & \frac{|x-y|}{3}+\frac{x}{6} \\ = \; & \lambda\left(|x-y|+\frac{x}{2}\right) \\ = \; & \lambda(d(x,y)+\varphi(x)+[\varphi(y)]) \\ = \; & \lambda(G(d(x,y),\varphi(x),\varphi(y))). \end{aligned} \tag{2.22}$$

The 3D graphs (plotted in MATLAB) in Figure 5 guarantee that the condition (2.22) holds.

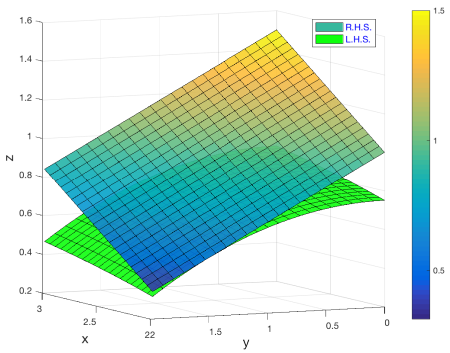

FIGURE 5. The graph of (2.22)

Considering all the above cases, we conclude that T satisfies the contractive condition (2.2). Therefore, by Theorem 2.2, T admits a unique φ-fixed point.

Corollary 2.1. *Let (X,d) be a complete metric space and $T : X \to X$ be a given mapping such that*

$$d(Tx,Ty) \leq \lambda(d(x,y))$$

for all $x, y \in X$, where $\lambda \in \Lambda$. Thus, T has a unique fixed point.

Proof. The proof of this corollary immediate by taking $\varphi \equiv 0$ in Theorem 2.2 and using condition $(G2)$. □

Remark 2.3. *Based on the fact that Λ is wider than Ψ, Corollary 2.1 is the real proper extension of the fixed point theorem of Boyd and Wong [5]. However, if we take $\varphi \equiv 0$ in Theorem 2.1 of Karapinar et al. [7], we have that the obtained result is equivalent to the Boyd and Wong fixed point theorem. This follows the advantage of our main result in this work, which is supported by several results in the literature, as shown in Figure 6.*

3. APPLICATIONS

In this section, we propose two applications which is derived from the main φ-fixed point result in the previous section. These applications involve the analysis of the homotopy result and the analysis of solutions to nonlinear Volterra integral equations of the second kind.

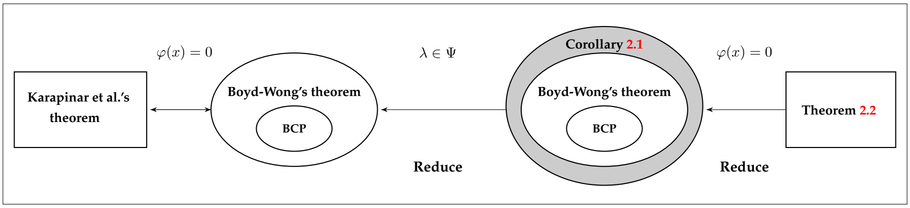

FIGURE 6. The difference of consequence between our theorem and Karapinar et al.'s theorem

3.1. **Application to the homotopy result.** In this part, we present the homotopic results that can be obtained from the φ-fixed point result in the previous section. We begin by introducing the new class of control functions used in the main result in this part. Denote by $\mathcal{G}^*$ the class of all functions $G \in \mathcal{G}$ satisfying the following property:

$(G4)$ for all $a, b, c, d \geq 0$,

$$a \leq c + d \implies G(a, b, 0) \leq G(c, b, 0) + d.$$

Example 3.4 ([7]). *Let $G_1, G_2, G_3 : [0, \infty)^3 \to [0, \infty)$ be defined by*

(i) $G_1(a, b, c) = (a + b)e^c$ *for all* $a, b, c \geq 0$;
(ii) $G_2(a, b, c) = (a + b)(c + 1)^n$ *for all* $a, b, c \geq 0$, *where* $n \in \mathbb{N}$;
(iii) $G_3(a, b, c) = ae^{c+b} + be^{a+c}$ *for all* $a, b, c \geq 0$.

Then, $G_1, G_2 \in \mathcal{G}^*$ and $\mathcal{G} \ni G_3 \notin \mathcal{G}^*$.

The following homotopy result can be derived from Theorem 2.2 and the same technique in the proof of Theorem 3.1 in [7].

Theorem 3.3. *Let (X, d) be a complete metric space, U be an open subset of X, and V be a closed subset of X with $U \subset V$. Suppose that $H : V \times [0, 1] \to X$ has the following properties:*

(C1) $x \neq H(x, \lambda)$ *for every* $x \in V \setminus U$ *and* $\lambda \in [0, 1]$;
(C2) *there exist a continuous function* $\varphi : X \to [0, \infty)$, $L \in (0, 1)$, *and* $G \in \mathcal{G}^*$ *such that*

$$G(d(H(x, \lambda), H(y, \lambda)), \varphi(H(x, \lambda)), \varphi(H(y, \lambda))) \leq LG(d(x, y), \varphi(x), \varphi(y))$$

for all $x, y \in V$ *and* $\lambda \in [0, 1]$;

(C3) *there exists a continuous function* $\eta : [0, 1] \to \mathbb{R}$ *such that*

$$G(d(H(x, \lambda), H(x, \mu)), \varphi(H(x, \lambda)), \varphi(H(x, \mu))) \leq |\eta(\lambda) - \eta(\mu)|$$

for all $x \in V$ *and* $\lambda, \mu \in [0, 1]$.

Then $H(\cdot, 0)$ has a fixed point if and only if $H(\cdot, 1)$ has a fixed point.

Based on the fact that the class $\mathcal{G}^*$ enlarges the class

$$\overline{\mathcal{G}} := \{G : [0, \infty)^3 \to [0, \infty) \mid G \text{ satisfies (G1), (G3), (G4) and } G \text{ is continuous}\}$$

which is defined in [7], we get the following result:

Corollary 3.2 (Theorem 3.1 in [7]). *Let (X, d) be a complete metric space, U be an open subset of X, and V be a closed subset of X with $U \subset V$. Suppose that $H : V \times [0, 1] \to X$ has the following properties:*

(C1) $x \neq H(x, \lambda)$ *for every* $x \in V \setminus U$ *and* $\lambda \in [0, 1]$;

(C2) *there exist a continuous function* $\varphi : X \to [0,\infty)$*,* $L \in (0,1)$*, and* $G \in \overline{\mathcal{G}}$ *such that*

$$G(d(H(x,\lambda),H(y,\lambda)),\varphi(H(x,\lambda)),\varphi(H(y,\lambda))) \le LG(d(x,y),\varphi(x),\varphi(y))$$

for all $x, y \in V$ *and* $\lambda \in [0,1]$*;*

(C3) *there exists a continuous function* $\eta : [0,1] \to \mathbb{R}$ *such that*

$$F(d(H(x,\lambda),H(x,\mu)),\varphi(H(x,\lambda)),\varphi(H(x,\mu))) \le |\eta(\lambda)-\eta(\mu)|$$

for all $x \in V$ *and* $\lambda, \mu \in [0,1]$*.*

Then $H(\cdot,0)$ *has a fixed point if and only if* $H(\cdot,1)$ *has a fixed point.*

3.2. **Application to the nonlinear Volterra integral equations.** In this part, we apply Theorem 2.2 to investigate the existence and uniqueness of a solution for the following nonlinear Volterra integral equation:

$$x(t) = \phi(t) + \int_a^t K(t,s,x(s))ds, \tag{3.23}$$

where $a, b \in \mathbb{R}$ with $a < b$, $x \in C[a,b]$ (the set of all continuous functions form $[a,b]$ into $\mathbb{R}$) is an unknown function, $\phi : [a,b] \to \mathbb{R}$ and $K : [a,b] \times [a,b] \times \mathbb{R} \to \mathbb{R}$ are two given continuous functions.

Theorem 3.4. *Consider the nonlinear Volterra equation (3.23). Suppose that there exists* $\lambda \in \Lambda$ *such that*

$$|K(t,s,r_1) - K(t,s,r_2)| \le \frac{\lambda(|r_1 - r_2|)}{b-a}$$

for all $t, s \in [a,b]$ *and* $r_1, r_2 \in \mathbb{R}$*. Then the nonlinear integral equation (3.23) has a unique solution.*

Proof. Let $X = C[a,b]$. Define the integral operator $T : X \to X$ for each $x \in X$ by a new function $Tx : [a,b] \to \mathbb{R}$ given by

$$(Tx)(t) = \phi(t) + \int_a^t K(t,s,x(s))ds$$

for all $t \in [a,b]$. We consider the complete metric space (X,d), where d is defined by

$$d(x,y) = \max_{t\in[a,b]} |x(t) - y(t)|$$

for each $x, y \in X$. Now, we define the functions $G : [0,\infty)^3 \to [0,\infty)$ and $\varphi : X \to [0,\infty)$ as follows:

$$G(a,b,c) = \max\{a,b\} + [c] \quad \text{for each} \quad a,b,c \in [0,\infty),$$

$$\varphi(x) = 0 \quad \text{for each} \quad x \in X.$$

Obviously, $G \in \mathcal{G}$ and $\lambda \in \Lambda$.

Next, we will show that (2.2) holds. Let $x, y \in X$ and $t \in [a, b]$. Then

$$\begin{aligned} |(Tx)(t) - (Ty)(t)| &= \left| \int_a^t K(t, s, x(s))ds - \int_a^t K(t, s, y(s))ds \right| \\ &= \left| \int_a^t (K(t, s, x(s)) - K(t, s, y(s)))ds \right| \\ &\leq \int_a^t |(K(t, s, x(s)) - K(t, s, y(s)))|ds \\ &\leq \frac{1}{b-a} \int_a^t \lambda(|x(s) - y(s)|)ds \\ &\leq \frac{1}{b-a} \int_a^t \lambda(d(x, y))ds \\ &\leq \frac{1}{b-a} \lambda(d(x, y))[b - a] \\ &= \lambda(d(x, y)). \end{aligned}$$

From the above inequality and by taking the maximum over t, we obtain

$$d(Tx, Ty) \leq \lambda(d(x, y)).$$

Consequently, it follows that

$$\max\{d(Tx, Ty), \varphi(Tx)\} + [\varphi(Ty)] \leq \lambda\big(\max\{d(x, y), \varphi(x)\} + [\varphi(y)]\big).$$

Hence,

$$G(d(Tx, Ty), \varphi(Tx), \varphi(Ty)) \leq \lambda\big(G(d(x, y), \varphi(x), \varphi(y))\big)$$

for all $x, y \in X$. Therefore, the inequality (2.2) together with all the assumptions of Theorem 2.2 are satisfied. Consequently, T has a unique fixed point, which implies that the integral equation (3.23) admits a unique solution in $C[a, b]$. □

4. Conclusion

In this work, we proved a φ-fixed point result for mappings satisfying the contractive condition involving control functions that do not have to be continuous and showed that our main results allowed us to find φ-fixed points of mappings in which the main results of Banach and Boyd-Wong cannot be applied, with Example 2.3. This claims the advantage of the main result of this work, with many known results from the past. Our results generalize the main φ-fixed point results of Karapinar et al. [7] and several known results in the literature. Actually, we can use the main φ-fixed point result in this paper to investigate the generalization of the fixed point result in the framework of partial metric spaces introduced by Karapinar et al. [7] (Corollary 3.3). This result will cover the partial metric version of the Boyd-Wong fixed point theorem and the fixed point result in partial metric spaces, as presented by Matthews [9].

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Research Article

Distrotion and quasisymmetric functions in quasiconformal mappings

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ABSTRACT. In this paper, we study the applications of special functions and quasisymmetry in quasiconformal mappings. Moreover, we estimate the distances between the image points of quasiconformal mappings under various metrics.

Keywords: Quasiconformal mapping, hyperbolic metric, distortion theorems.

2020 Mathematics Subject Classification: 30C65, 51M10.

1. INTRODUCTION

In the two-dimensional setting, the foundations of quasiconformal mapping theory were laid by H. Grötzsch in 1928, followed by significant contributions from O. Teichmüller during the period 1938–1944, and later by L. Bers and L. V. Ahlfors beginning in the early 1950s. The extension of this theory to higher dimensions (n-dimensional spaces) was initiated by F. W. Gehring and J. Väisälä in the early 1960s. Conformal invariants, such as the modulus of a family of curves, play a central role in the study of quasiconformal mappings. These invariants can often be characterized using specific conformal mappings. In the context of hyperbolic geometry on the unit disk, hyperbolic lines are defined as geodesicscurves that minimize length with respect to the hyperbolic metric. While this metric serves as a powerful tool for analyzing various planar domains, its extension to proper subdomains of higher-dimensional Euclidean spaces $\mathbb{R}^n$, with $n \geq 3$, is generally not feasible.

Classical Function Theory (CFT) extensively utilizes the hyperbolic metric due to its invariance under conformal mappings, particularly Möbius transformations. This invariance often renders theoretical results more naturally and effectively expressed in hyperbolic terms than in Euclidean geometry. A notable example is the Schwarz lemma, which characterizes analytic self-maps of the unit disk as contractions with respect to the hyperbolic metric. Similarly, Nevanlinnas principle underscores the fundamental role of hyperbolic geometry in the analytic structure of CFT. While classical tools such as power series expansions and integral representations are inherently local and do not capture global conformal invariance, the method of extremal lengthdeveloped by Ahlfors and Beurlingpreserves this invariance and has become a powerful and essential technique in modern CFT. However, generalizing these concepts to higher-dimensional settings ($n \geq 3$) poses significant difficulties, primarily due to the breakdown of complex analytic tools such as multiplication of complex numbers and the failure of

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the Riemann mapping theorem in higher dimensions. Liouvilles theorem further illustrates the rigidity of conformal mappings in higher dimensions, where they reduce to Möbius transformations, thus limiting the scope of conformal theory. Consequently, extending classical function theoretic methods to higher-dimensional spaces requires fundamentally different approaches and presents deep analytical challenges [26].

In this paper, we present a concise overview of quasiconformal mappings, various metric structures, and special functions, with a focus on their applications in deriving distortion theorems for quasiconformal mappings. The study is grounded in the foundational and recent developments presented in [3, 6, 7, 9, 10, 13, 17, 20, 24, 26, 22].

The structure of the paper is as follows. Section 1 provides an introduction to the topic. Section 2 presents the necessary definitions and explores the interrelations among various metric. In Section 3, we introduce certain special functions along with their associated inequalities, formulated as lemmas, and discuss the notion of quasi-symmetry. Section 4 offers a concise account of Moris theorem and the Schwarz lemma, including some of their recent advancements. Finally, Section 5 contains key lemmas and distortion theorems formulated with respect to different metrics.

2. Definitions and metrics

In our study, the following metric spaces will be central to our discussion:

(1) the Euclidean space $\mathbb{R}^n$,
(2) the Poincaré half-space $\mathbb{H}^n = \mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_n > 0\}$,
(3) the Möbius space $\overline{\mathbb{R}}^n = \mathbb{R}^n \cup \{\infty\}$.

2.1. **Notations.** For $x \in \mathbb{R}^n$ and $r > 0$ let

$$\mathbb{B}^n(x,r) = \{z \in \mathbb{R}^n : |x-z| < r\},$$
$$S^{n-1}(x,r) = \{z \in \mathbb{R}^n : |x-z| = r\}$$

denote the ball and sphere, respectively, centred at x with radius r. Moreover, we denote $\mathbb{B}^n(r) = \mathbb{B}^n(0,r)$, $S^{n-1}(r) = S^{n-1}(0,r)$, $\mathbb{B}^n(1) = \mathbb{B}^n$, $S^{n-1}(1) = S^{n-1}$. We denote by $e_1, e_2, \ldots, e_n$ the standard unit vectors in $\mathbb{R}^n$.

2.2. **Metric space (X, d).** A metric on a non-empty set X is a real-valued function $d : X \times X \to [0,\infty)$ satisfying the following properties:

(1) $d(x,y) \geq 0$ and $d(x,x) = 0, \quad \forall x, y \in X$,
(2) $d(x,y) = d(y,x), \forall x, y \in X$,
(3) $d(x,z) \leq d(x,y) + d(y,z), \quad \forall x, y, z \in X$.

The metric space (X, d) is a set X equipped with a metric d on X. For instance,

(1) the set of real numbers $\mathbb{R}$, with the usual distance function $d(x,y) = |x-y|$, $x, y \in \mathbb{R}$ is a metric space.
(2) More generally $(\mathbb{R}^n, |\cdot|)$ is a metric space.
(3) If (X_j, d_j), $j = 1, 2$ are metric spaces and $f : (X_1, d_1) \to (X_2, d_2)$ is an injection, then $m_f(x,y) = d_2(f(x), f(y))$ is a metric.
(4) If (X, d) is a metric space, then also (X, d^a) is a metric space for all $a \in (0, 1]$.

2.3. **Isometry.** Let (X, d_1) and (Y, d_2) be metric space and let $f : X \to Y$ be homeomorphism. Then f is an isometry if

$$d_2\left(f(x), f(y)\right) = d_1(x,y)$$

for all $x, y \in X$.

2.4. **Lipschitz mappings.** Let (X, d_1) and (Y, d_2) be metric space and let $f : X \to Y$ be continuous and let $L \geq 1$. We say that f is L-Lipschitz if

$$d_2(f(x), f(y)) \leq L d_1(x, y)$$

for all $x, y \in X$. In addition, if f is homeomorphism and

$$d_1(x, y)/L \leq d_2(f(x), f(y)) \leq L d_1(x, y)$$

for all $x, y \in X$, we say that f is L-Lipschitz.

2.5. **Möbius transformation.** The following types of mappings generate the group of Möbius transformation.

(1) A reflection in hyperplane $P(a, t)$:

$$f_1(x) = x - 2(xa - t)\frac{a}{|a|^2},$$

where $P(a, t) = \{x \in \mathbb{R}^n : x \cdot a = t\}$, $t \in \mathbb{R}$, $a \in \mathbb{R}^n$.

(2) An inversion in $\mathbb{S}^{n-1}(a, r)$:

$$f_2(x) = a + \frac{r^2(x - a)}{|x - a|^2}, \quad f_2(a) = \infty,\ f_2(\infty) = a.$$

If $G \subset \overline{\mathbb{R}}^n$, we denote by $\mathcal{GM}(G)$ the group of all Möbius transformations with $f(G) = G$. For the further information and results on Möbius transformation the reader is referred to Beardon's book [5, p.32]. If f is an inversion in $S^{n-1}(a, r)$, then the following identity

$$|f(x) - f(y)| = \frac{r^2|x - y|}{|x - a||y - a|}$$

holds for $x, y \in \mathbb{R}^n \backslash \{a\}$, see [24, (1.5)].

Let $f(u) = r^2 u/|u|^2$, $r > 0$, $u \in \mathbb{R}^n \setminus \{0\}$ and let $z = x\,(|x| + |x - y|)\,/|x|$ for all $x, y \in \mathbb{R}^n \setminus \{0\}$ with $|x| \leq |y|$. Then

$$|f(x) - f(z)| \leq |f(x) - f(y)| \leq 3|f(x) - f(z)|.$$

Equality holds in the upper bound for $x = -y$ (see [7] for proof).

For $a \in \mathbb{B}^n$, define $T_a : \mathbb{B}^n \to \mathbb{B}^n$ as

$$T_a(x) = p_a \circ \sigma_a\,,$$

where p_a is the reflection in $(n-1)$-dimensional plane through the origin and orthogonal to a, and σ_a is the inversion in the sphere $S^{n-1}(a/|a|, \sqrt{1/|a|^2 - 1})$. For $n = 2$, we have

$$T_a(z) = \frac{z - a}{1 - \bar{a}z},$$

see [13, p.11, p.459].

2.6. **Absolute (cross) ratio.** For distinct points $a, b, c, d \in \overline{\mathbb{R}}^n$, the absolute (cross) ratio is defined by

$$|a, b, c, d| = \frac{q(a, c)q(b, d)}{q(a, b)q(c, d)},$$

where for all $x, y \in \mathbb{R}^n$

$$\begin{cases} q(x, y) = \frac{|x-y|}{\sqrt{1+|x|^2}\sqrt{1+|y|^2}}, \\ q(x, \infty) = \frac{1}{\sqrt{1+|x|^2}}, \end{cases}$$

is the spherical (chordal) metric [24]. The chordal metric is estimated as follows

$$\frac{|x-y|}{|1+(|x|)||(1+|y|)|} \le q(x,y) \le \frac{2|x-y|}{|1+(|x|)||(1+|y|)|}\,,$$

for all $x, y \in \mathbb{R}^n$ [24, p.5]. If $q(f(x), f(y)) = q(x,y)$ for all $x, y \in \overline{\mathbb{R}}^n$ and $f \in \mathcal{GM}(\overline{\mathbb{R}}^n)$, then f is called spherical isometry. One of the most important properties of Möbius transformations is that they preserve absolute (cross) ratio, i.e.,

$$|f(a), f(b), f(c), f(d)| = |a, b, c, d|$$

for all $a, b, c, d \in \overline{\mathbb{R}}^n$ and $f \in \mathcal{GM}$.

2.7. **Conformal mapping.** Let D and D' be domains in $\mathbb{R}^n$ and $f : D \to D'$ be a homeomorphism. We call f *conformal* if $f \in \mathcal{C}^1$, $J_f(x) \neq 0$ for all $x \in D$, $|f'(x)h| = |f'(x)||h|$ for all $x \in D$ and $h \in \mathbb{R}^n$. If D and D' are domains in $\overline{\mathbb{R}}^n$, we call a homeomorphism $f : D \to D'$ conformal if the restriction of f to $D \setminus \{\infty, f^{-1}(\infty)\}$ is conformal.

For $n = 2$ there are many conformal mappings, i.e., Riemann mapping Theorem, Schwarz-Christoffel formula. For $n \ge 3$ conformal maps are Liouville's theorem, Möbius transformations. Therefore conformal invariance for the space $n \ge 3$ is very different from the plane case $n = 2$ [26].

2.8. **Quasiconformal mappings.** Given domains $D, D' \in \overline{\mathbb{R}}^n$. Let $f : D \to D'$ be a homeomorphism. If Γ is a family of curves in D, then $\Gamma' = f(\Gamma)$. We set

$$K_1(f) = \sup \frac{M(\Gamma')}{M(\Gamma)}, K_0(f) = \sup \frac{M(\Gamma)}{M(\Gamma')}\,,$$

where the suprema is taken over is taken over all curve families Γ in D such that $M(\Gamma)$ and $M(\Gamma')$ are not simultaneously 0 or ∞. We call $K_1(f)$ the inner dilatation, $K_0(f)$ the outer dilatation, and $K(f) = \max\{K_1(f), K_0(f)\}$ the maximum dilatation of f. If $K(f) \le K < \infty$, then f is known as K-quasiconformal. If $K(f) < \infty$, then f is quasiconformal. This geometric definition is due to Väisälä [22, p.42].

In particular, a 1-quasiconformal $f : \overline{\mathbb{R}}^n \to \overline{\mathbb{R}}^n$ is a Möbius transformation.

A set G is connected if for all $x, y \in G$ there exists a path $\gamma : [0,1] \to G$ such that $\gamma(0) = x$ and $\gamma(1) = y$. Sometimes, we write Γ_{xy} for the set of all paths joining x with y in G.

2.9. **Inner metric of a set.** Let $G \subset X$. For fixed $x, y \in X$ the inner metric with respect to G is defined by

$$d(x,y) = \inf\{\ell(\gamma) : \gamma \in \Gamma_{xy}\,, \gamma \in G\}.$$

2.10. **Geodesics.** A path $\gamma : [0,1] \to G$ where G is a domain, is a geodesic joining $\gamma(0)$ and $\gamma(1)$ if $\ell(\gamma) = d(\gamma(0), \gamma(1))$ and

$$\ell(\gamma) = d(\gamma(0), \gamma(t)) + d(\gamma(t) + \gamma(1)), \quad t \in (0,1).$$

For example, the segment

$$[x,y] = \{z \in \mathbb{R}^n : z = tx + (1-t)y, t \in [0,1]\}$$

is a geodesic in $(\mathbb{R}^n, |\cdot|)$.

2.11. **Path integrals.** For a locally rectifiable path $\gamma : \Delta \to X$ and a continuous function $f : \gamma\Delta \to [0, \infty]$, the path integral is defined in two steps as follows:

(1) If γ is rectifiable, then
$$\int_\gamma f ds = \int_0^{\ell(\gamma)} f(\gamma^o(t))|(\gamma^o)'(t)|dt,$$
where γ^o is the normal representation of a rectifiable path.

(2) If γ is locally rectifiable, then we set
$$\int_\gamma f ds = \sup\left\{\int_\alpha \,:\, \ell(\alpha) < \infty,\, \alpha \text{ is a subpath of } \gamma)\right\}.$$

2.12. **Weighted length.** Let $G \subset X$ be a domain and $w : G \to (0, \infty)$ continuous. For fixed $x, y \in D$, we define
$$d_w(x, y) = \inf\left\{\ell_w(\gamma) = \int_\gamma w(\gamma(z))|dz| : \gamma \in \Gamma_{xy},\, \ell(\gamma) < \infty\right\}.$$
One can see easily that d_w defines a metric on G and (G, d_w) is a metric space.

2.13. **Hyperbolic metric.** Let Γ be a family of all rectifiable curves in $\mathbb{B}^n$ or $\mathbb{H}^n$, joining x and y. Then the hyperbolic metric ρ in unit ball $\mathbb{B}^n$ and in $\mathbb{H}^n$ is defined by
$$\rho_{\mathbb{B}^n} = \inf_{\gamma\in\Gamma} \int_\gamma \frac{2|dz|}{1 - |z|^2}, \quad x, y \in \mathbb{B}^n \quad \text{and} \quad \rho_{\mathbb{H}^n} = \inf_{\gamma\in\Gamma} \int_\gamma \frac{|dz|}{d(z, \partial\mathbb{H}^n)}, \quad x, y \in \mathbb{H}^n,$$
respectively.

If D is a simply connected domain of the extended complex plane $\mathbb{C}$ and $\text{card}\partial D > 1$, then by the Riemann mapping theorem there exists a conformal mapping f from D onto the unit disk $\{z \in C : |z| < 1\}$.

The hyperbolic metric ρ in D is defined by the following formula
$$\rho_D(x, y) = \inf_{\gamma\in\Gamma} \int_\gamma \eta_D(z)|dz|, \quad \text{where} \quad \eta_D(z) = \frac{2|f'(z)|}{1 - |f(z)|^2},$$
where Γ is the family of all rectifiable curves in D joining x and y. The following inequalities give us an upper and lower bound for η_D
$$\frac{1}{4d(z, \partial D)} \le \eta_D(z) \le \frac{1}{d(z, \partial D)}.$$
Once the Euclidean distances $x_n = d(x, \partial\mathbb{H}^n)$, $y_n = d(x, \partial\mathbb{H}^n)$ and $|x - y|$ are known then the hyperbolic distance $\rho_{\mathbb{H}^n}$ can be determined by the following formula
$$\cosh(\rho(x, y)) = 1 + \frac{|x - y|^2}{2x_n y_n}, \quad x, y \in \mathbb{H}^n.$$
The counterpart of the above formula for $\mathbb{B}^n$ is given by
$$\sinh^2\left(\frac{1}{2}\rho(x, y)\right) = \frac{|x - y|^2}{(1 - |x|^2)(1 - |y|^2)}, \quad x, y \in \mathbb{B}^n,$$
see [5, p.35-40]. For the unit ball $\mathbb{B}^n$, we have two more definition for the hyperbolic metric given below
$$\rho_{\mathbb{B}^n}(x, y) = \sup\{\log|a, x, y, b|\}, \quad x, y \in \mathbb{B}^n,\, a, b \in \partial\mathbb{B}^n,$$
$$\rho_{\mathbb{B}^n}(x, y) = \log|x_*, x, y, y_*|, \quad x, y \in \mathbb{B}^n.$$

If $h \in \mathcal{GM}$ and $x, y \in \mathbb{B}^n$, then

$$\rho(h(x), h(y)) = \rho(x, y), \quad x, y \in \mathbb{B}^n.$$

2.14. **Metrics j and $\tilde{j}$.** Let (X, d) be a metric space and $G \subset X$ be an open set with non-empty boundary. Then for all $x, y \in G$, the following formulas

$$j_G(x, y) = \log\left(1 + \frac{|x-y|}{\min\{d(x), d(y)\}}\right)$$

and

$$\tilde{j}_G(x, y) = \log\left(1 + \frac{|x-y|}{d(x)}\right) + \log\left(1 + \frac{|x-y|}{d(y)}\right)$$

define metrics on G (see [12, 24]), where $d(x) = d(x, \partial G) = \inf\{|x-z| : z \in \partial G\}$ is the distance between a point $x \in G$ and the boundary ∂G of G. In literature, metrics j and $\tilde{j}$ are known as distance ratio metric and ratio metric, respectively.

The following inequalities show the relation between $j, \tilde{j}$ and ρ metrics in different domains [24, p.29],

(1)
$$j_G(x, y) \le \tilde{j}_G(x, y) \le 2j_G(x, y) \quad \text{for all } x, y \in G \subset \mathbb{R}^n,$$

(2)
$$j_{\mathbb{B}^n}(x, y) \le \rho_{\mathbb{B}^n}(x, y) \le 4j_{\mathbb{B}^n}(x, y) \quad \text{for all } x, y \in \mathbb{B}^n,$$

(3)
$$j_{\mathbb{H}^n}(x, y) \le \rho_{\mathbb{H}^n}(x, y) \le 2j_{\mathbb{H}^n}(x, y) \quad \text{for all } x, y \in \mathbb{H}^n.$$

2.15. **Quasihyperbolic metric.** Let $G \subset \mathbb{R}^n$ be a domain. The quasihyperbolic metric k_G is defined by

$$k_G(x, y) = \inf_{\gamma \in \Gamma} \int_\gamma \frac{|dz|}{d(z, \partial G)}, \quad x, y \in G,$$

where Γ is the family of all rectifiable curves in G joining x and y [12]. In the following inequalities, we show the relation of k_G metric with other metrics [12, 16, 24, 26],

(1)
$$\frac{1}{2}\rho_{\mathbb{B}^n}(x, y) \le k_{\mathbb{B}^n}(x, y) \le \rho_{\mathbb{B}^n}(x, y) \quad \text{for all } x, y \in \mathbb{B}^n,$$

(2)
$$j_{\mathbb{B}^n}(x, y) \le k_{\mathbb{B}^n}(x, y) \le (1+r)j_{\mathbb{B}^n}(x, y) \quad \text{for all } x, y \in \mathbb{B}^n(r),\ r \in (0,1),$$

(3)
$$j_G(x, y) \le \log\left(1 + \frac{l}{\min\{d(x), d(y)\}}\right) \le k_G(x, y) \quad \text{for all } x, y \in G \subset \mathbb{R}^n,$$

where $l = \inf\{\ell(\gamma) : \gamma \in \Gamma_{x,y}\}$,

(4)
$$j_{\mathbb{B}^n}(0, y) = k_{\mathbb{B}^n}(0, y) = \log\frac{1}{1-|x|} \quad \text{for all } x \in \mathbb{B}^n,$$

(5)
$$j_{\mathbb{B}^n}(ax, ay) = k_{\mathbb{B}^n}(ax, ay) = \frac{1-x}{1-y}, \quad a \in S^{n-1},$$

where $x, y \in (0,1)$ with $x < y$.

2.16. **Ferrand's metric.** Let $D \subset \overline{\mathbb{R}}^n$ be a domain with card$\partial D \geq 2$. Then τ_D defines a metric in D, defined

$$\sigma_D(x,y) = \inf_{\gamma \in \Gamma} \int_\gamma w_D(z)|dz|,$$

where Γ is the family of all rectifiable curves in D joining x and y, and

$$w_D = \sup_{a,b \in \partial D} \frac{|a-b|}{|z-a||z-b|}.$$

2.17. **Apollonian metric.** For a proper subdomain D of $\overline{\mathbb{R}}^n$, the Apollonian metric is defined by

$$\alpha_D(x,y) = \sup_{a,b \in \partial D} \log |a,x,y,b|, \quad x,y \in D.$$

2.18. **Absolute ratio metric.** Let G be an open subset of $\overline{\mathbb{R}}^n$ with card$\partial G \geq 2$. Then for all $x, y \in G$ the following formula

$$\delta_G(x,y) = \log\left(1 + \sup_{a,b \in \partial G} |a,x,y,b|\right),$$

defines a metric in G ([20]).

3. Special functions

3.1. **Modulus of curve family.** Let Γ be a family of curves in $\overline{\mathbb{R}}^n$. We denote by $F(\Gamma)$ the set of all non-negative Borel functions $\rho : \overline{\mathbb{R}}^n \to \mathbb{R}^n \cup \{\infty\}$ such that $\int_\gamma \rho ds \geq 1$ for every locally rectifiable curves $\gamma \in \Gamma$. We define the modulus

$$M(\Gamma) = \inf_{\rho \in F(\Gamma)} \int_{\mathbb{R}^n} \rho^n dm,$$

where m stands for the n-dimensional Lebesgue measure.

For sets $E, F \subset G$, and $G \subset \overline{\mathbb{R}}^n$, we denote by $\Delta(E,F;G)$ the curve family of all curves joining the sets E and F in G, and let $\Delta(E,F) = \Delta(E,F;\overline{\mathbb{R}}^n)$. The capacity of a ring $R(E,F)$ is cap$R(E,F) = M(\Delta(E,F))$, where M is modulus of family of curves $\Delta(E,F)$.

3.2. **Conformal invariants.** Let G be a proper subdomain of $\overline{\mathbb{R}}^n$. For $x, y \in G$ with $x \neq y$ we define

$$\lambda_G(x,y) = \inf_{C_x, C_y} M(\Delta(C_x, C_y; G)),$$

where $C_z = \gamma_z[0,1]$ and $\gamma_z : [0,1) \to G$ is a curve such that $\gamma_z(0) = z$ and $\gamma_z(t) \to \partial G$ when $t \to 1$, $z = x, y$.

If G is a proper subdomain of $\overline{\mathbb{R}}^n$. Then for $x, y \in G$, we define

$$\mu_G(x,y) = \inf_{C_{xy}} M(\Delta(C_{xy}, \partial G; G),$$

where the infimum is taken over all continua C_{xy} such that $C_{xy} = \gamma[0,1]$ and γ is a curve with $\gamma(0) = x$ and $\gamma(1) = y$.

Conformal invariant λ_G was introduced by Ferrand [25]. Teichmüller studied λ_G for $n = 2$ and $G = \mathbb{R}^2 \setminus \{0\}$, while μ_G was studied by Grötzsch for $n = 2$ and $G = \mathbb{B}^2$, see [1]. It was remarked in [3, p.320] that $\lambda_G^{1/(1-n)}$ and μ_G are metrics.

For all $x, y \in D$ with $x \neq y$ and $d \in \{\mathbb{H}^n, \mathbb{B}^n\}$, the following relations hold true [14]

$$\lambda_D(x,y) = 2^{-n}\tau_n(\cosh(\rho_D(x,y)/2)),$$

$$\mu_D(x,y) = \tau_n(1/\tanh(\rho_D(x,y)/2)).$$

It was proved in [14] that metrics μ_{H^2}, μ_{B^2} and $\lambda_{B^2}^{-1}$ are not Hölder continuous with respect to the hyperbolic metric.

3.3. **Rings.** A domain G in $\overline{\mathbb{R}}^n$ is called a *ring* if $\overline{\mathbb{R}}^n \setminus G$ has two components. If the components are C_0 and C_1 we write $G = R(C_0, C_1)$. For $s > 1$, the complementary components of the Grötzsch ring $R_{G,n}$ in $\mathbb{R}^n$ are $\overline{\mathbb{B}}^n$ and $[se_1, \infty]$. For $t > 0$, the complementary components of the Teichmüller ring $R_{T,n}$ in $\mathbb{R}^n$ are $[-e_1, 0]$ and $[te_1, \infty]$.

The Grötzsch ring constant λ_n is defined by

$$\log \lambda_n = \lim_{r \to 0+} (M_n(r) + \log r),$$

where $M_n(r)$ is the conformal modulus of the Grötzsch ring [3, p.167].

The capacities of $T_{G,n}$ and $R_{G,n}(s)$ are denoted by decreasing homeomorphism functions $\tau_n : (0,\infty) \to (0,\infty)$ and $\gamma_n : (1,\infty) \to (0,\infty)$, respectively, with the following formula [13, p.121]

$$\tau_n(s) = M(\Delta([-e_1,0],[se_1,\infty];\mathbb{R}^n)), \quad s > 0,$$
$$\gamma_n(s) = M(\Delta(\overline{\mathbb{B}}^n,[se_1,\infty];\mathbb{R}^n), \quad s > 1.$$

Moreover, the functions τ_n and γ_n satisfying the following functional identity

$$\gamma_n(t) = 2^{n-1}\tau_n(t^2 - 1), \quad t > 1,$$

[24, Lemma 5.53].

Utilizing the capacities τ_n and γ_n of $T_{G,n}$ and $R_{G,n}(s)$, respectively, the following identities were proved in [24, Thm 8.6],

$$\mu_{\mathbb{B}^n}(x,y) = \gamma_n\left(\frac{1}{\tanh(\rho(x,y)/2)}\right),$$
$$\lambda_{\mathbb{B}^n}(x,y) = \frac{\tau_n}{2}\left(\tanh^2(\rho(x,y)/2)\right)$$

for all $x, y \in \mathbb{B}^n$.

3.4. **Elliptic integrals.** The plane Grötzsch ring can be mapped onto an annulus by an elliptic function [5]

$$\tau_2(s) = \frac{2\pi}{\mu(1/s)}, \quad s > 1,$$

where

$$\mu(r) = \frac{\pi}{2}\frac{K\sqrt{1-r^2}}{\mathcal{K}(r)}, \quad \mathcal{K}(r) = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-r^2t^2)}}.$$

The function $\mathcal{K}(r)$ is called a complete elliptic integral of the first kind. The function $\mu(r)$ satisfies the following functional identities

$$\mu(r) = 2\mu\left(\frac{2\sqrt{r}}{1+r}\right), \quad \mu(r)\mu\left(\frac{1-r}{1+r}\right) = \frac{\pi^2}{2}, \quad \mu(r)\mu(\sqrt{1-r^2}) = \frac{\pi^2}{4}.$$

For $0 < r < 1$, the function $\mu(r)$ is estimated as below [24, p.67]

$$\log\left(\frac{1}{r}\right) < \mu(r) < \log\left(\frac{4}{r}\right).$$

3.5. **Distortion function.** For all $K > 0$ and $n \geq 2$, the distortion function $\varphi_{K,n} : [0,1] \to [0,1]$ is defined by

$$\varphi_{K,n}(r) = \frac{1}{\gamma_n^{-1}(K\gamma_n(1/t))}, \quad 0 < t < 1,$$

$\varphi_{K,n}(0) = 0$ and $\varphi_{K,n}(1) = 1$.

For later use we fix the constants α and β as follows

$$\alpha = K^{1/(1-n)} = \beta, \quad K \geq 1\,.$$

Lemma 3.1. *For $n \geq 2$, $K \geq 1$, and $r \in [0,1]$, we have*

$$\begin{cases}\varphi_{K,n}(r) \leq \lambda_n^{1-\alpha} r^{\alpha}, \quad \varphi_{1/K,n}(r) \geq \lambda_n^{1-\beta} r^{\beta}\end{cases} \tag{3.1}$$

and

$$\begin{cases}\lambda_n^{1-\alpha} \leq 2^{1-\alpha} K \leq 2^{1-1/K} K, \quad \lambda_n^{1-\beta} \geq 2^{1-\beta} K^{-\beta} \geq 2^{1-K} K^{-K}\end{cases} \tag{3.2}$$

see [24, Thm 7.47] *and* [3, Lem 875].

For $n \geq 2$, $t \in (0,\infty)$, $K > 0$, we denote

$$\eta_{K,n}(t) = \tau_n^{-1}\left(\frac{1}{K}\tau_n(t)\right) = \frac{1 - \varphi_{1/K,n}(1/\sqrt{1+t})^2}{\varphi_{1/K,n}(1/\sqrt{1+t})^2}\,.$$

For $n \geq 2$, $1 \leq K < \infty, t \in [0,\infty)$, let

$$\begin{aligned}\eta^*_{K,n}(t) = \sup\{&|f(x)| : |x| \leq t,\ f \in QC_K(\overline{\mathbb{R}}^n),\\ &f(0) = 0,\ f(e_1) = e_1,\ f(\infty) = \infty\},\end{aligned}$$

where $QC_K(\mathbb{B}^n)$ denotes the set of all K-quasiconformal maps of $\overline{\mathbb{R}}^n$ into itself. It is well know that $\eta^*_{K,2}(t) = \eta_{K,2}(t)$, [17, p.80].

For $n \geq 2$ and $0 < r < 1$,

$$\varphi^*_{K,n}(r) = \begin{cases}\sup\{|f(x)| : |x| = r, f \in QC_K(\mathbb{B}^n), f(0) = 0\} & \text{if } 1 \leq K < \infty\\ \inf\{|f(x)| : |x| = r, f \in QC_{1/K}(\mathbb{B}^n), f(0) = 0\} & \text{if } 0 < K \leq 1.\end{cases}$$

Lemma 3.2. *For $n \geq 2$, $K \geq 1$, and $alpha = K^{1/(1-n)} = 1/\beta$, $0 < r < 1$, we have* [3, 13(3)]

$$r^{\alpha} \leq \varphi^*_{K,n}(r) \leq \varphi_{K,n}(r) \leq \left(\frac{\lambda_n}{2}\right)^{1-\alpha} (1+r')^{1-\alpha} r^{\alpha} \leq \lambda_n^{1-\alpha} r^{\alpha}\,.$$

Lemma 3.3. *In the following inequalities we have upper bounds for $\eta^*_{K,n}$* [3, Thm 14.6 & 14.8]. *For $n \geq 2$ and $K \geq 1$, we have*

$$\begin{cases}\eta^*_{K,n}(t) \leq \eta^*_{K,n}(1)\varphi^*_{K,n}(t), & 0 \leq t \leq 1,\\ \eta^*_{K,n}(t) \leq \eta^*_{K,n}(1)\varphi^*_{1/K,n}(t), & t \geq 1,\\ \eta^*_{K,n}(1) \leq e^{4K(K+1)\sqrt{K-1}}\,. \end{cases}$$

3.6. **Quasisymmetic function.** Let $\eta : [0,\infty) \to [0,\infty)$ be a homeomorphism with $\eta(0) = 0$. A homeomorphism between two metric spaces X and Y with distances denoted by $|x-y|$ is called η-quasisymmetric if

$$\frac{|f(x)-f(y)|}{|f(x)-f(z)|} \leq \eta\left(\frac{|x-y|}{|x-z|}\right)$$

for all distinct points $x, y, z \in D$ with $x \neq z$ (see [21]).

The quasisymmetry can be extended to maps $\overline{\mathbb{R}}^n$ by using absolute (cross) ratio

$$|f(a), f(b), f(c), f(d)| \leq \eta(|a, b, c, d|)$$

for all distinct points $a, b, c, d \in \overline{\mathbb{R}}^n$. Mappings satisfying this condition are called quasi-möbius mappings [23]. A K-quasiconformal mapping $f : \mathbb{R}^2 \to \mathbb{R}^2$ is $\eta^*_{K,2}$-quasisymmetric [3, Thm 9.42].

4. Mori's theorem and Schwarz Lemma

In 1956, A. Mori [18] proved the following theorem.

Theorem 4.1. *Let $f : \mathbb{B}^2 \to \mathbb{B}^2$ be a K-quasiconformal mapping with $f(0) = 0$ and $f(\mathbb{B}^2) = \mathbb{B}^2$. Then for all $x, y \in \mathbb{B}^2$*

$$|f(x) - f(y)| \leq 16|x - y|^K .$$

The number 16 cannot be replace by an smaller absolute constant.

In [17, p.68], it was conjectured that the best constant on place of 16 is $16^{1-1/K}$. In [3, Thm 15.4], it is proved that

$$|f(x) - f(y)| \leq 64^{1-1/K}|x - y|^K ,$$

where f is as in Theorem 4.1.

In 1997, Qiu [19] prove the best constant $46^{1-1/K}$ in place of 16. For $n \geq 2$, Fehlmann and Vuorinen [9] proved the following theorem.

Theorem 4.2. *Let $f : \mathbb{B}^n \to \mathbb{B}^n$ be a K-quasiconformal mapping with $f(0) = 0$ and $f(\mathbb{B}^n) = \mathbb{B}^n$. Then for all $x, y \in \mathbb{B}^n$*

$$|f(x) - f(y)| \leq M(n, K)|x - y|^\alpha , \quad \alpha = K^{1/(1-n)},$$

where the constant $M(n, K)$ has the following properties:

(1) $M(n, K) \to 1$ as $K \to 1$, uniformly in n,
(2) $M(n, K)$ remains bounded for fixed K and varying n,
(3) $M(n, K) \leq 3\lambda_n^2$ for all $K \geq 1$.

With the same assumption as in Theorem 4.2, Anderson and Vamanamurthy [2] proved that $M(n, K) \leq 4\lambda_n^{1(1-\alpha)}$, where $\alpha = K^{1/(1-n)}$, and $\lambda \in [4, 2e^{n-1})$ is the Grötzsch ring constant [24, p.89].

In 2011, Bhayo and Vuorinen [6] proved that for $n \geq 2,\ K \geq 1$,

$$M(n, K) \leq T(n,\ K) \leq \inf\{h(t) : t \geq 1\},$$

where $\alpha = K^{1/(1-n)} = 1/\beta$. There exists a number $K_1 > 1$ such that for all $K \in (1, K_1)$ the function h has a minumum at a point $t_1 > 1$ and

$$T(n, K) \leq h(t_1) = \frac{3^{1-\alpha}(\beta - \alpha)^{\alpha^2}}{\alpha^\alpha}\lambda_n^{\alpha-\alpha^2} + \lambda_n^{\beta-1}\left(\frac{3\alpha\lambda_n^{\alpha-1}}{(\beta - \alpha)^\alpha}\right)^{\eta-\alpha} .$$

Moreover, for $\beta \in (1, 2)$, we have

$$h(t_1) = 3^{\beta-\alpha}2^{1-\alpha}K^5\left(\frac{3}{2}\sqrt[4]{\beta - \alpha} + \exp(\beta - 1)\right).$$

In particular, $h(t_1) \to 1$ when $K \to 1$. For $K \in (1, 1.3044)$ and $K > 8.9105$ the bound $T(n, K)$ improves the bound $M(n, K)$ given in [2].

A counterpart of Theorem 4.2 was given as follows with some restrictions.

Theorem 4.3. *For $n \geq 2$, $K \geq 1$, let $f : \overline{\mathbb{R}}^n \to \overline{\mathbb{R}}^n$ be a K-quasiconformal mapping with $f(0) = 0$, $f(e_1) = e_1$, and $f(\infty) = \infty$. Then*

$$|f(x) - f(y)| \leq c(K)^2 |x - y|^\beta$$

for all $x \in \mathbb{R}^n \setminus \mathbb{B}^n$, $y \in \mathbb{R}^n$ with $1 < |x| < |x - y|$, and

$$|f(x) - f(y)| \leq c(K)^2 |x - y|^\alpha$$

for all $x \in \mathbb{B}^n$, $y \in \mathbb{R}^n$ with $|x - y| < |x| < 1$, where $c(K) = 2^{K-1} K^K e^{4K(K+1)\sqrt{K-1}}$.

Theorem 4.4 ([24, 11.2]). *Let $f : \mathbb{B}^n \to \mathbb{R}^n$ be a nonconstant K-quasiregular mapping [22] with $f(\mathbb{B}^n) \subset \mathbb{B}^n$ and $\alpha = K^{1/(1-n)}$. Then for all $x, y \in \mathbb{B}^n$*

$$\tanh\left(\frac{\rho(f(x), f(y))}{2}\right) \leq \varphi_{K,n}\left(\tanh\frac{\rho(x,y)}{2}\right) \leq \lambda_n^{1-\alpha}\left(\tanh\frac{\rho(x,y)}{2}\right).$$

If $f(0) = 0$, then for all $x \in \mathbb{B}^n$

$$|f(x)| \leq \lambda_n^{1-\alpha} |x|^\alpha .$$

For the plane case $n = 2$, the quasiconformal version of the above theorem appears in [17, p.65]. The following formula

$$\rho(f(x), f(x)) \leq K(\rho(x,y) + \log 4)$$

was proved in [8, Thm 5.1], where f is as in Theorem 4.4.

Under the same assumption as in Theorem 4.4, if $f(0) = 0$ then for all $x \in \mathbb{B}^n$

$$\varphi_{1/K,n}(|x|) \leq |f(x)| \leq \varphi_{K,n}(|x|),$$

[6, Lem 6.1].

For $n \geq 2$, $\alpha = K^{1/(1-n)}$, let $f : \overline{\mathbb{R}}^n \to \overline{\mathbb{R}}^n$ be a K-quasiconforma mapping fixing $0, e_1$, and ∞, then

$$|f(x)| \leq K \lambda_n^{2(1-\alpha)} |x|^\alpha$$

for all $x \in \mathbb{R}^n \setminus \{0\}$,$\mathbb{B}^n$ with $0 < |x| \leq (K-1)/K$.

Theorem 4.5 ([6]). *Let $f : \mathbb{B}^2 \to \mathbb{R}^2$ be a nonconstant K-quasiregular mapping [22] with $f(\mathbb{B}^2) \subset \mathbb{B}^2$, then for all $x, y \in \mathbb{B}^2$*

$$\rho(f(x), f(y)) \leq c(K) \max\{\rho(x,y), \rho(x,y)^{1/K}\},$$

where $c(K) = 2\text{arctanh}(\varphi(\tanh(1/2)))$.

5. Quadruples and distortion theorems

Lemma 5.4 ([3, Thm 14.7]). *For $K \geq 1$ let f be a K-quaisconformal automorphism of the plane $\overline{\mathbb{R}}^2$. Then*

$$\frac{1}{\lambda(K)} \min\{t^{1/K}, t^K\} \leq |f(a), f(b), f(c), f(d)| \leq \lambda(K) \max\{t^{1/K}, t^K\},$$

for each ordered quadruple of distinct points a, b, c, d in the plane, where $t = |a, b, c, d|$. Moreover the inequalities are sharp for each $K \geq 1$.

Lemma 5.5. *For $K \geq 1$ let $f : \overline{\mathbb{R}}^2 \to \overline{\mathbb{R}}^2$ be K-quasiconformal. Then*

$$|f(a), f(b), f(c), f(d)| \leq e^{\pi K(K-1)} \max\{t^{1/K}, t^K\},$$

for each ordered quadruple of distinct points $a, b, c, d \in \mathbb{R}^2$.

Lemma 5.6 ([3, Thm 14.8]). *Let $f : \overline{\mathbb{R}}^n \to \overline{\mathbb{R}}^n$ be K-quasiconformal. Then*

$$\frac{1}{\eta^*_{K,n}}(1/|a,b,c,d|) \le |f(a),f(b),f(c),f(d)| \le \eta^*_{K,n}(|a,b,c,d|)\,,$$

for each ordered quadruple of distinct points $a,b,c,d \in \overline{\mathbb{R}}^n$.

Theorem 5.6 ([3, Thm 15.20]). *Let $G \subset \mathbb{R}^n$ and let $f : G \to G' \subset \overline{\mathbb{R}}^n$ be a K-quasiconformal. Then*

$$q(f(x),f(y))q(\partial G') \le 128\left(\frac{|x-y|}{d(x,\partial G)}\right)^{1/K},$$

where q is a chordal metric.

Theorem 5.7. *Let $G \subset \mathbb{R}^n$, $f : \overline{\mathbb{R}}^n \to \overline{\mathbb{R}}^n$ be K-quasiconformal. Then*

$$\delta_{fG}(f(x),f(y)) \le c(K)\beta \max\{\delta_G(x,y), \delta_G(x,y)^\alpha\},$$

where $c(K) = 2^{K-1}K^K e^{4(K+1)\sqrt{K-1}}$ and $\alpha = K^{1/(1-n)} = 1/\beta$.

Theorem 5.8 ([20]). *Let $f : \overline{\mathbb{R}}^n \to \overline{\mathbb{R}}^n$ be a K-quasiconformal, G and fG are open sets of $\overline{\mathbb{R}}^n$ with card $\partial G \ge 2$, then*

$$\delta_{fG}(f(x),f(y)) \le \eta_{K,n}(\delta_G(x,y))$$

for all $x,y \in G$.

Theorem 5.9 ([20]). *Let $f : G \to G'$ be a K-quasiconformal mapping, where G and $fG = G'$ are proper subsets of $\overline{\mathbb{R}}^n$, then*

$$\frac{\mu_G(x,y)}{K} \le \mu'_G(f(x),f(y)) \le K\mu_G(x,y)$$

for all $x,y \in G$. If card $\partial G \ge 2$, *then*

$$\frac{\lambda_G(x,y)}{K} \le \lambda'_G(f(x),f(y)) \le K\lambda_G(x,y)$$

for all $x,y \in G$.

Theorem 5.10. *Let $G \subset \mathbb{R}^2$ and $f : \overline{\mathbb{R}}^2 \to \overline{\mathbb{R}}^2$ be a K-quasiconformal mapping with $f(\infty) = \infty$, then*

$$\tilde{j}_{fG}(f(z_1),f(z_2)) \le e^{\pi(K-1/K)}K \max\{\tilde{j}_G(x,y), \tilde{j}_G(x,y)\}$$

for all $z_1, z_2 \in G$.

If G is a disk or half plane then $\rho_G(z_1,z_2) \le \tilde{j}_G(z_1,z_2)$, for all $z_1,z_2 \in G$ ([11]).

Theorem 5.11 ([15]). *Let $G = \mathbb{R}^n \setminus \{0\}$, and $f : \overline{\mathbb{R}}^n \to \overline{\mathbb{R}}^n$ be a K-quasiconformal mapping with $f(0) = 0$ and $f(\infty) = \infty$, then*

$$j_{fG}(f(x),f(y)) \le a(K)\max\{j_G(x,y), j_G(x,y)\}$$

for all $x,y \in G$, where $\alpha = K^{1/(1-n)} = 1/\beta$, $a(K) = e^{60\sqrt{K-1}}$, and $a(K) \to 1$ as $K \to 1$.

The above theorem is refined as follows:

Theorem 5.12. *Let $G = \mathbb{R}^n \setminus \{0\}$, and $f : \overline{\mathbb{R}}^n \to \overline{\mathbb{R}}^n$ be a K-quasiconformal mapping with $f(0) = 0$ and $f(\infty) = \infty$, then*

$$j_{fG}(f(x),f(y)) \le b(K)\max\{j_G(x,y), j_G(x,y)\}$$

for all $x,y \in G$, where $\alpha = K^{1/(1-n)} = 1/\beta$, $b(K) = 2^{K-1}K^K e^{(4K(K+1))\sqrt{K-1}}$, and $a(K) \to 1$ as $K \to 1$.

In [7] a two exponent variant of the function $x \mapsto |x|^{p-1}x$ was defined for $a, b > 0, x \in \mathbb{R}^n$,

$$\mathcal{A}_{a,b}(x) = \begin{cases} |x|^{a-1}x & if\ |x| < 1 \\ |x|^{b-1}x & if\ |x| \geq 1. \end{cases}$$

For $a = b$, the function $\mathcal{A}_{a,b}$ defines a quasiconformal mapping and it has been used in many examples to illuminate various properties of these maps [22, p.49]. For instance, if $a \in (0,1)$ the function $\mathcal{A}_{a,b}$ is Hölder-continuous at the origin.

The next theorem tells us how the hyperbolic distances from the origin are changed under the radial selfmapping of the the unit disk, $z \mapsto |z|^{1/K-1}z, K > 1$, which is the restriction of $\mathcal{A}_{1/K,1/K}(z)$ to the unit disk, see [6].

Theorem 5.13. *The following inequality holds for $K \geq 1$, $|z| < 1$;*

$$\rho(0, \mathcal{A}_{1/K,K}(z)) \leq K \max\{\rho(0,|z|), \rho^{1/K}(0,|z|)\},$$

where ρ is the hyperbolic metric [24, p. 19].

Corollary 5.1 ([7]).

(1) *Let $D = \mathbb{R}^n \setminus \{0\}$, then we have*

$$j_D(\mathcal{A}_{1/K,K}(x), \mathcal{A}_{1/K,K}(y)) \leq 2^{1-1/K} \max\{j_D(x,y), j_D^{1/K}(x,y)\}$$

for all $K \geq 1, x, y \in \mathbb{B}^n \cap D$.

(2) *The following inequality holds for $K \geq 1$:*

$$||x|^{K-1}x - |y|^{K-1}y| \leq e^{\pi(K-1/K)}|x|^{K-1/K} \max\{|x-y|^{1/K}, |x-y|^K\}$$

for all $x, y \in \mathbb{C} \setminus \overline{\mathbb{B}}^2$.

(3) *The following inequality holds for $K \geq 1$ and for all $x, y \in \mathbb{R}^n \setminus \overline{\mathbb{B}}^n$:*

$$||x|^{\beta-1}x - |y|^{\beta-1}y| \leq c(K)|x|^{\beta-\alpha} \max\{|x-y|^{\alpha}, |x-y|^{\beta}\},$$

where $c(K) = 2^{K-1}K^K \exp(4K(K+1)\sqrt{K-1})$ and $\alpha = K^{1/(1-n)} = 1/\beta$.

Corollary 5.2 ([7]). *The following inequalities hold for $K \geq 1$;*

$$\left| \frac{x}{|x|^{1+1/K}} - \frac{y}{|y|^{1+1/K}} \right| \leq 2^{1-1/K} \frac{|x-y|^{1/K}}{(|x||y|)^{1/K}} \tag{5.3}$$

for all $x, y \in \mathbb{R}^n \setminus \mathbb{B}^n$,

$$\left| \frac{x}{|x|^{1+\beta}} - \frac{y}{|y|^{1+\beta}} \right| \leq \frac{c(K)}{|x|^{\beta-\alpha}} \max\left\{ \left(\frac{|x-y|}{|x||y|}\right)^{\alpha}, \left(\frac{|x-y|}{|x||y|}\right)^{\beta} \right\} \tag{5.4}$$

for all $x, y \in \mathbb{B}^n$,

$$\left| \frac{x}{|x|^{1+K}} - \frac{y}{|y|^{1+K}} \right| \leq \frac{e^{\pi(K-1/K)}}{|x|^{K-1/K}} \max\left\{ \left(\frac{|x-y|}{|x||y|}\right)^{1/K}, \left(\frac{|x-y|}{|x||y|}\right)^{K} \right\} \tag{5.5}$$

for all $x, y \in \mathbb{B}^2$.

The proofs of the theorems presented in this section are primarily based on the definition of quasisymmetry (see 3.6) and Lemmas 3.1, 3.2, 3.3, 5.4, 5.5, 5.6. For details, the reader is referred to [7, 6, 4, 20, 3].

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Research Article

Mappings contracting perimeters of triangles in perturbed metric spaces

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ABSTRACT. In the present paper, we introduce the notion of mappings contracting perimeters of triangles in perturbed metric spaces, which we call perturbed mappings contracting perimeters of triangles. We provide a fixed point result for such mappings. We illustrate that our results are more general with some examples.

Keywords: Fixed point, mappings contracting triangles, perturbed metric spaces.

2020 Mathematics Subject Classification: 47H10, 54H25.

1. INTRODUCTION AND PRELIMINARIES

Since Banach's pioneering fixed-point result was introduced (see [1]), there has been a sustained effort within the research community to derive more general results. There are two main directions of research: one is to relax the contractive conditions (see, for example, [3, 4, 6, 10, 13, 15, 16, 19, 22, 20] and the references therein), another area of interest has been altering the topological structure in which fixed-point results are established (see, for instance, [5, 7, 11, 12, 21], and the references therein).

In the present paper, we introduce the notion of mappings contracting perimeters of triangles in perturbed metric spaces, which we call perturbed mappings contracting perimeters of triangles. The new results combine the ideas of perturbed metric structures, recently proposed by Jleli and Samet in [8], with mappings contracting perimeters of triangles introduced by Petrov in [16].

We establish a fixed point theorem for such mappings in complete perturbed metric spaces, showing that the existence of fixed points can be guaranteed under a suitable contractive condition involving a three point condition. To highlight the applicability and generality of our approach, we provide an example demonstrating that our fixed point result remains valid in situations where classical contractive conditions fail. The framework proposed here opens new directions for further exploration of fixed point theory in generalized and perturbed metric spaces.

Recently, Jleli and Samet introduced a novel framework in [8], termed *perturbed metric spaces*, which extends the traditional concept of a metric space as follows:

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Definition 1.1. *Let $D, P : X \times X \to [0, \infty)$ be two given mappings. D is a perturbed metric on X with respect to P if the function*

$$d = D - P : X \times X \to \mathbb{R}, \quad (x, y) \mapsto D(x, y) - P(x, y)$$

is a metric on X. This means that for all $x, y, z \in X$, the following conditions hold:

(i) $(D - P)(x, y) \geq 0$,
(ii) $(D - P)(x, y) = 0$ *if and only if* $x = y$,
(iii) $(D - P)(x, y) = (D - P)(y, x)$,
(iv) $(D - P)(x, y) \leq (D - P)(x, z) + (D - P)(z, y)$.

We call P a perturbed mapping, $d = D - P$ the exact metric, and (X, D, P) a perturbed metric space.

Definition 1.2. *Let (X, D, P) be a perturbed metric space, $\{z_n\}$ a sequence in X, and $T : X \to X$.*

(i) *We say that $\{z_n\}$ is a perturbed convergent sequence in (X, D, P) if $\{z_n\}$ is convergent in the metric space (X, d), where $d = D - P$ is the exact metric.*
(ii) *We say that $\{z_n\}$ is a perturbed Cauchy sequence in (X, D, P) if $\{z_n\}$ is a Cauchy sequence in the metric space (X, d).*
(iii) *We say that (X, D, P) is a complete perturbed metric space if (X, d) is a complete metric space, or equivalently, if every perturbed Cauchy sequence in (X, D, P) is a perturbed convergent sequence in (X, D, P).*
(iv) *We say that T is a perturbed continuous mapping if T is continuous with respect to the exact metric d.*

In [8], a generalization of Banach's fixed point theorem is proved:

Theorem 1.1. *Let (X, D, P) be a complete perturbed metric space and $T : X \to X$ a given mapping. Assume that the following conditions hold:*

(i) *T is a perturbed continuous mapping.*
(ii) *There exists $\lambda \in (0, 1)$ such that*

$$D(Tx, Ty) \leq \lambda D(x, y) \quad \text{for all } x, y \in X.$$

Then, T admits a unique fixed point.

Very recently, Petrov introduced in [16] a new type of mappings called mappings contracting perimeters of triangles, which are a three-point analogue of Banach contractions [1]:

Definition 1.3 (Petrov [16])**.** *Let (X, d) be a metric space with $|X| \geq 3$. We shall say that $T\colon X \to X$ is a mapping contracting perimeters of triangles on X if there exists $\alpha \in [0, 1)$ such that the inequality*

$$d(Tx, Ty) + d(Ty, Tz) + d(Tz, Tx) \leq \alpha[d(x, y) + d(y, z) + d(z, x)],$$

holds for all three pairwise distinct points $x, y, z \in X$.

Petrov proved in [16] a fixed point theorem for this kind of mapping:

Theorem 1.2 (Petrov [16])**.** *Let (X, d), $|X| \geq 3$ be a complete metric space and let $T\colon X \to X$ be a mapping contracting perimeters of triangles on X. Then, T has a fixed point if and only if T does not possess periodic points of prime period 2. The number of fixed points is at most 2.*

The newly introduced mappings were further studied and extended in [2, 9, 14, 17, 18, 19, 15, 24].

2. Main results

Definition 2.4. *Let (X, D, P) be a perturbed metric space with $|X| \geq 3$. We shall say that $T\colon X \to X$ is a perturbed mapping contracting perimeters of triangles if there exists $\lambda \in [0, 1)$ such that the inequality*

$$D(Tx, Ty) + D(Ty, Tz) + D(Tz, Tx) \leq \lambda[D(x, y) + D(y, z) + D(z, x)], \tag{2.1}$$

holds for all three pairwise distinct points $x, y, z \in X$.

Theorem 2.3. *Let (X, D, P) be a complete perturbed metric space and $T : X \to X$ such that*
 (i) *$T(Tx) \neq x$ for all $x \in X$ such that $Tx \neq x$.*
 (ii) *T is a perturbed continuous mapping contracting perimeters of triangles.*

Then, T has a fixed point. The number of fixed points is at most two.

Proof. Let $x_0 \in X$, arbitrarily chosen, but fixed and the Picard iteration

$$x_{n+1} = Tx_n, \quad \forall n \geq 0.$$

We shall show that T has at least one fixed point. Suppose that x_n is not a fixed point of the mapping T for every $n = 0, 1, \ldots$ Then, we have $x_n = Tx_{n-1} \neq x_{n-1}$ and $x_{n+1} = T(Tx_{n-1}) \neq x_{n-1}$ for every $n = 1, 2, \ldots$. Hence, by condition (i), x_{n-1}, x_n and x_{n+1} are pairwise distinct.

Let

$$d_n = D(x_n, x_{n+1}) + D(x_{n+1}, x_{n+2}) + D(x_{n+2}, x_n).$$

Since x_{n-1}, x_n and x_{n+2} are pairwise distinct, by (2.1) we have

$$D(x_{n+1}, x_{n+2}) + D(x_{n+2}, x_{n+3}) + D(x_{n+3}, x_{n+1}) \leq \alpha[D(x_n, x_{n+1}) + D(x_{n+1}, x_{n+2}) + D(x_{n+2}, x_n)],$$

i.e.

$$d_{n+1} \leq \alpha d_n,$$

and inductively we obtain

$$d_{n+1} \leq \alpha^{n+1} d_0.$$

It is clear that $D(x_{n+1}, x_{n+2}) \leq d_n$, so we obtain

$$D(x_{n+1}, x_{n+2}) \leq d_n \leq \alpha^n d_0.$$

Now, let $d = D - P$ be the exact metric. Then, from the above inequity, we obtain that

$$d(x_n, x_{n+1}) + P(x_n, x_{n+1}) \leq \alpha^n d_0, \quad \forall n \geq 0.$$

Since $d(x_n, x_{n+1}) \leq d(x_n, x_{n+1}) + P(x_n, x_{n+1})$, we get that

$$d(x_n, x_{n+1}) \leq \alpha^n d_0, \quad \forall n \geq 0.$$

Then,

$$\begin{aligned} d(x_n, x_{n+p}) &\leq \alpha^n d_0 + \alpha^{n+1} d_0 + \cdots + \alpha^{n+p-1} d_0 \\ &\leq \frac{\alpha^n}{1 - \alpha} d_0. \end{aligned}$$

Since $\alpha \in [0, 1)$ we obtain that $\{x_n\}$ is a Cauchy sequence in the metric space (X, d), so $\{x_n\}$ is a perturbed Cauchy sequence in (X, D, P). Thus, by completeness of the perturbed metric space (X, D, P), there exists $x^* \in X$ such that

$$\lim_{n\to\infty} d(x_n, x^*) = 0.$$

To show that $Tx^* = x^*$, since T is a perturbed continuous mapping, it follows that

$$\lim_{n\to\infty} d(Tx_n, Tx^*) = 0,$$

which implies

$$\lim_{n\to\infty} d(x_{n+1}, Tx^*) = 0.$$

Since $d = D - P$ is a metric on X, by uniqueness of the limit we obtain $x^* = Tx^*$, i.e. x^* is a fixed point of T.

Finally, if there exist at least three pairwise distinct fixed points x^*, y^* and z^*, then $Tx^* = x^*$, $Ty^* = y^*$ and $Tz^* = z^*$, which contradicts (2.1). Therefore, the number of fixed points is at most two.

□

Let us provide an example of perturbed mappings that contracting triangles.

Example 2.1. *Let $X = \{0, 1, 2\}$, and define the mappings: Let $X = \{0, 1, 2\}$, and define:*

$$d(x, y) = \begin{cases} 1, & \text{if } x \neq y, \\ 0, & \text{if } x = y, \end{cases} \qquad P(x, y) = \begin{cases} 0.1, & \text{if } \{x, y\} = \{1, 2\}, \\ 0, & \text{otherwise}. \end{cases}$$

Let $D(x, y) = d(x, y) + P(x, y)$.

(X, D, P) is a complete perturbed metric space since (X, d) is complete.

Define the mapping $T : X \to X$ by

$$T(0) = 0, \quad T(1) = 0, \quad T(2) = 2.$$

We have

$$\begin{aligned} D(T0, T1) + D(T1, T2) + D(T2, T0) &= D(0, 0) + D(0, 2) + D(2, 0) \\ &= 0 + 1.1 + 1.1 = 2.2, \\ D(0, 1) + D(1, 2) + D(2, 0) &= 1 + 1.1 + 1 = 3.1. \end{aligned}$$

Hence,

$$D(Tx, Ty) + D(Ty, Tz) + D(Tz, Tx) \leq \lambda[D(x, y) + D(y, z) + D(z, x)]$$

with $\lambda = \frac{2.2}{3.1} \approx 0.71 < 1$. Thus, T is a perturbed mapping contracting triangles.

However, T is not a Banach since

$$D(T0, T2) = D(0, 2) = 1.1 = D(0, 2).$$

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Research Article

Propagation of solitons and nonlinear behavior in nonlinear power law fibers

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ABSTRACT. This study investigates soliton propagation within the framework of nonlinear optics, specifically under the influence of a detuning parameter modeled by the complex GinzburgLandau equation (CGLE). Employing the φ^6-model expansion method, we derive a diverse set of analytical solutions, including trigonometric, hyperbolic, and rational function solutions. Notably, singular soliton solutions are obtained and shown to exhibit positive characteristics. The analysis is conducted in the context of nonlinear optical fibers governed by a power-law nonlinearity. The results contribute to a deeper understanding of the nonlinear dynamical behavior inherent in the CGLE and highlight the effectiveness of the applied method as a robust and efficient tool for obtaining reliable solutions to a wide class of nonlinear partial differential equations. To illustrate the physical features of the obtained solutions, several representative results are visualized through two-dimensional, three-dimensional, and contour plots.

Keywords: φ^6-model expansion approach, the complex Ginzburg–Landau equation, traveling wave solution, power law nonlinearity.

2020 Mathematics Subject Classification: 35Qxx, 35Exx.

1. INTRODUCTION

Optical solitons represent one of the most rapidly advancing areas of research in optoelectronics and nanoelectronics. A growing body of literature regularly reports new findings on this topic, reflecting its increasing significance. Various mathematical models have been developed to describe the propagation dynamics of solitons in optical fibers, including the nonlinear Schrödinger equation, the Schrödinger-Hirota equation, the Sasa-Satsuma equation, the Manakov equation, the Biswas-Milovic equation, and several others. Among these, the Complex GinzburgLandau Equation (CGLE) stands out as a particularly prominent model for capturing soliton behavior. Over the past few decades, numerous analytical and numerical methods have been devised to study the evolution of nonlinear systems across diverse physical domains-from fluid dynamics to optical wave propagation [20, 24, 8, 9, 22, 12, 18, 10, 25, 13, 5, 26, 14]. Analytical solitary wave solutions play a crucial role in understanding and analyzing long-distance signal transmission in optical communication systems, especially over transcontinental and trans-oceanic spans. Recent studies further underscore the importance of optical solitons in various branches of photonics, particularly in nonlinear optics and spectroscopy [23, 19, 21, 11, 6, 7].

In recent years, the Complex GinzburgLandau Equation (CGLE) with power-law nonlinearity has attracted significant attention from researchers. Mirzazadeh et al. [4] employed the

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semi-inverse variational principle to derive certain trivial solutions to the equation. Arnous, Ahmed H. et al. [2] applied the modified simple equation method to obtain bright and singular soliton solutions. Arshed [3] utilized the exp(-Φ(ξ))-expansion method and reported several distinct types of soliton solutions. Mirzazadeh, Mohammad et al. [28] adopted multiple integration schemes to extract a variety of soliton solutions, while Al-Ghafri [1] employed the Weierstrass elliptic function approach along with hyperbolic function solutions to obtain bright and singular solitons. In the present study, the CGLE with power-law nonlinearity is investigated using the recently developed φ^6-model expansion method [15, 17, 29, 27, 16]. This approach enables the recovery of optical solitary wave solutions, further demonstrating the methods effectiveness in handling complex nonlinear systems.

The paper is structured as follows. Section 2 outlines the mathematical formulation and theoretical analysis of the model. In Section 3, we provide a detailed description of the φ^6-model expansion method, highlighting its methodology and underlying principles. Section 4 applies this approach to the complex GinzburgLandau Equation (CGLE) with power-law nonlinearity, successfully deriving a variety of novel traveling wave solutions. The physical characteristics of these solutions are visualized through three-dimensional and density plots, and a brief discussion on the dynamical behavior of the obtained solitons is also presented. Finally, Section 5 offers concluding remarks and highlights the broader implications of the results.

2. Mathematical analysis of the model

In this study, we adopt the dimensionless formulation of the (GCLE) as presented in [2, 27], which serves as the basis for our subsequent analysis

$$iq_t + aq_{xx} + bF(|q|^2)q = \frac{1}{|q|^2 q^\star}\left[\alpha |q|^2 (|q|^2)_{xx} - \beta \left\{(|q|^2)_x\right\}^2\right] + \gamma q. \tag{2.1}$$

Here, q denotes a complex-valued function representing the wave profile observed in various physical contexts, including nonlinear optics and plasma physics. The variable x corresponds to the non-dimensional spatial coordinate along the fiber, and t denotes the dimensionless time. The parameters a, b, α, β and γ are real-valued constants, where a and b are associated with the Group Velocity Dispersion (GVD) and the nonlinear coefficient, respectively. The coefficients α, β and γ arise from perturbative effects, particularly detuning.

In Eq. (2.1), F denotes a smooth, real-valued algebraic function, and the term $F(|q|^2)q$ is assumed to be k-times continuously differentiable. These smoothness and differentiability requirements ensure the well-posedness of the model and facilitate the application of analytical techniques to obtain exact or approximate solutions, implying that

$$F(|q|^2)q \in \cup_{n,m=1}^{\infty} C^k\left((-m,m)\times(-n,n);R^2\right). \tag{2.2}$$

By setting up

$$\alpha = 2\beta. \tag{2.3}$$

Eq. (2.1) turns to

$$iq_t + aq_{xx} + bF(|q|^2)q = \frac{\beta}{|q|^2 q^\star}\left[2 |q|^2 (|q|^2)_{xx} - \left\{(|q|^2)_x\right\}^2\right] + \gamma q. \tag{2.4}$$

The solution of Eq. (2.1) proceeds via the standard decomposition into amplitude and phase components

$$q(x,t) = P(\psi)e^{i(-kx+wt+\theta)} \tag{2.5}$$

and the wave variable ψ is expressed as

(2.6) $$\psi = \lambda (x - vt).$$

Here, the function P characterizes the pulse shape, while v denotes the solitons velocity. In the phase factor, k corresponds to the soliton frequency, ω to the soliton wave number, and θ to the phase constant. Inserting the amplitudephase decomposition into Eq. (2.4) and separating the result into real and imaginary parts yields the following pair of equations:

(2.7) $$-\left(ak^2 + \gamma + \omega\right) P + bF\left(P^2\right) P + \lambda^2 (a - 4\beta) P^{''} = 0$$

and

(2.8) $$v = -2ka.$$

In the part 4 of this paper, Eq. (2.7) will be examined using power law of nonlinearity.

3. Description of the method

Based on the work in [5, 15, 17, 29, 27, 16], the essential steps of the recently proposed φ^6-model expansion technique are given below:

Step-1: We consider the Nonlinear Evolution Equation (NLEE) given for $q = q(x,t)$ below

(3.9) $$H(q, q_x, q_t, q_{xx}, q_{xt}, ...) = 0.$$

Here, H denotes a polynomial in $q(x,t)$ and its highest-order partial derivatives, which also includes the nonlinear components.

Step-2: Utilizing the wave transformation

(3.10) $$q(x,t) = q(\psi), \quad \psi = x - vt.$$

Here, v represents the wave speed, allowing Eq. (3.9) to be rewritten as the following nonlinear ordinary differential equation.

(3.11) $$\Omega(q, q^{'}, qq^{'}, q^{''}, ...) = 0,$$

where primes denote derivatives with respect to ψ.

Step-3: Let us suppose that Eq. (3.11) admits a formal solution:

(3.12) $$q(\psi) = \sum_{i=0}^{2N} \alpha_i U^i(\psi),$$

where $\alpha_i (i = 0, 1, 2, \ldots, N)$ represent undetermined constants, N is determined via the balancing method, and $U(\psi)$ is a solution of the auxiliary Nonlinear Ordinary Differential Equation (NLODE);

(3.13) $$U'^2(\psi) = h_0 + h_2 U^2(\psi) + h_4 U^4(\psi) + h_6 U^6(\psi),$$
$$U^{''}(\psi) = h^2 U(\psi) + 2h_4 U^3(\psi) + 3h_6 U^5(\psi),$$

here, the real constants $h_i($ with $i = 0, 2, 4, 6)$ will be identified in subsequent steps.

Step-4: The solution to Eq. (3.13) is commonly known to be;

(3.14) $$U(\psi) = \frac{P(\psi)}{\sqrt{fP^2(\psi) + g}},$$

provided that $0 < fP^2(\psi) + g$ and $P(\psi)$ is the Jacobi elliptic equation solution

(3.15) $$P'^2(\psi) = l_0 + l_2 P^2(\psi) + l_4 P^4(\psi),$$

here, the unknown constants l_i (with $i = 0, 2, 4$) are to be determined, while f and g are expressed as

$$f = \frac{h_4(l_2 - h_2)}{(l_2 - h_2)^2 + 3l_0 l_4 - 2l_2(l_2 - h_2)}, \tag{3.16}$$
$$g = \frac{3l_0 h_4}{(l_2 - h_2)^2 + 3l_0 l_4 - 2l_2(l_2 - h_2)},$$

under the restriction condition

$$h_4^2(l_2 - h_2)[9l_0 l_4 - (l_2 - h_2)(2l_2 + h_2)] + 3h_6[-l_2^2 + h_2^2 + 3l_0 l_4]^2 = 0. \tag{3.17}$$

Step-5: As discussed in [5], the Jacobi elliptic function solutions of Eq. (3.15) are well-established for values of ρ in the interval $0 < \rho < 1$. By substituting these solutions, along with Eq. (3.14), into the solution ansatz given in Eq. (3.12), we obtain exact analytical solutions to Eq. (3.9).

Function	$\rho \to 1$	$\rho \to 0$	Function	$\rho \to 1$	$\rho \to 0$
$sn(\psi, \rho)$	$tanh(\psi)$	$sin(\psi)$	$ds(\psi, \rho)$	$csch(\psi)$	$csc(\psi)$
$cn(\psi, \rho)$	$sech(\psi)$	$cos(\psi)$	$sc(\psi, \rho)$	$sinh(\psi)$	$tan(\psi)$
$dn(\psi, \rho)$	$sech(\psi)$	1	$sd(\psi, \rho)$	$sinh(\psi)$	$sin(\psi)$
$ns(\psi, \rho)$	$coth(\psi)$	$csc(\psi)$	$nc(\psi, \rho)$	$cosh(\psi)$	$sec(\psi)$
$cs(\psi, \rho)$	$csch(\psi)$	$cot(\psi)$	$cd(\psi, \rho)$	1	$cos(\psi)$

TABLE 1. Jacobi elliptic functions

4. IMPLEMENTATION OF THE φ^6-MODEL EXPANSION METHOD

Power-law nonlinearity arises naturally in a variety of physical systems and is particularly prominent in materials such as semiconductors, doped glasses, and photorefractive media, where the nonlinear response of the refractive index depends on the intensity of the optical field in a non-quadratic manner. This type of nonlinearity is widely regarded as a natural generalization of the well-known Kerr law (which corresponds to the case $n = 1$), offering a more flexible and physically realistic framework for modeling nonlinear optical phenomena. In this context, the nonlinear function takes the form $F(u) = u^n$, allowing for a broader range of nonlinear behaviors depending on the value of the exponent n. Under this assumption, Eq. (2.4) reduces to a generalized version of the Complex GinzburgLandau Equation (CGLE) that accommodates higher or lowernorder nonlinearities, making it applicable to a wider class of optical media beyond those governed by cubic nonlinearities. This generalization is not only mathematically significant but also physically meaningful, as it enables the modeling of soliton propagation in engineered materials where the nonlinear response deviates from standard Kerr-type behavior [4, 2, 3, 28]. The parameter n thus plays a critical role in tuning the strength and nature of the nonlinearity, influencing the existence, stability, and dynamical properties of the resulting soliton solutions. In this case, Eq. (2.4) reduces to

$$iq_t + (b\,|q|^{2n})q + aq_{xx} = \frac{\beta}{|q|^2\, q^\star}\left[2\,|q|^2\,(|q|^2)_{xx} - \left\{(|q|^2)_x\right\}^2\right] + \gamma q, \tag{4.18}$$

and Eq. (2.7) is transformed

$$-\left(ak^2 + \gamma + \omega\right)P + bP^{2n+1} + \lambda^2\,(a - 4\beta)\,P'' = 0. \tag{4.19}$$

Here in Eq. (4.18) the parameter n dictates the power law nonlinearity. For stability issues, it is necessary to have $0 < n < 2$, and in particular $n \neq 2$, to avoid self-focusing singularity.

Setting

$$P = p^{\frac{1}{n}},$$

Eq. (4.19) transform to

$$-n^2\left(ak^2+\gamma+\omega\right)p^2 + n^2bp^4 + \lambda^2(a-4\beta)\left[(1-n)\left(p^{'}\right)^2 + npp^{''}\right] = 0. \tag{4.20}$$

In Eq. (4.20), we get $N = 1$ by balancing $pp^{''}$ with P^4, we obtain the following by substituting $N = 1$ in Eq. (3.12).

$$P(\psi) = \alpha_0 + \alpha_1 U(\psi) + \alpha_2 U^2(\psi), \tag{4.21}$$

where α_0, α_1 and α_2 are constants to be determined.

By substituting the solution ansatz given in Eq. (4.21) and the auxiliary ordinary differential equation Eq. (3.13) into Eq. (4.19), and collecting terms of each power of $U^i(\psi)$, $i = 0, 1, \ldots, 8$, we obtain a system of algebraic equations by setting the coefficients of like powers to zero.

$$\begin{aligned}
U^0(\psi) &: -n^2\left(ak^2+\gamma+\omega\right)\alpha_0^2 + bn^2\alpha_0^4 \\
&\quad - \lambda^2(-1+n)(a-4\beta)h_0\alpha_1^2 + 2\lambda^2 n(a-4\beta)h_0\alpha_0\alpha_2 = 0, \\
U^1(\psi) &: -n\alpha_0\alpha_1\left(2n\left(ak^2+\gamma+\omega\right) - \lambda^2(a-4\beta)h_2\right) \\
&\quad + 4bn^2\alpha_0^3\alpha_1 - 2\lambda^2\alpha_1(-2+n)(a-4\beta)h_0\alpha_2 = 0, \\
U^2(\psi) &: -\left(-\lambda^2(a-4\beta)h_2 + n^2\left(ak^2+\gamma+\omega-6b\alpha_0^2\right)\right)\alpha_1^2 \\
&\quad - 2n\alpha_0\left(-2\lambda^2(a-4\beta)h_2 + n\left(ak^2+\gamma+\omega-2b\alpha_0^2\right)\right)\alpha_2 \\
&\quad - 2\lambda^2(-2+n)(a-4\beta)h_0\alpha_2^2 = 0, \\
U^3(\psi) &: \alpha_1\left(2\lambda^2 n(a-4\beta)h_4\alpha_0 + 4bn^2\alpha_0\alpha_1^2\right) \\
&\quad + \alpha_1\alpha_2\left(\lambda^2(4+n)(a-4\beta)h_2 - 2n^2\left(ak^2+\gamma+\omega-6b\alpha_0^2\right)\right) = 0, \\
U^4(\psi) &: bn^2\alpha_1^4 + 12bn^2\alpha_0\alpha_1^2\alpha_2 - \left(-4\lambda^2(a-4\beta)h_2 + n^2\left(ak^2+\gamma+\omega-6b\alpha_0^2\right)\right)\alpha_2^2 \\
&\quad + \lambda^2(a-4\beta)h_4\left((1+n)\alpha_1^2 + 6n\alpha_0\alpha_2\right) = 0, \\
U^5(\psi) &: 3\lambda^2 n(a-4\beta)h_6\alpha_0\alpha_1 + 4\alpha_1\alpha_2\left(\lambda^2(1+n)(a-4\beta)h_4 + bn^2\left(\alpha_1^2+3\alpha_0\alpha_2\right)\right) = 0, \\
U^6(\psi) &: \lambda^2(a-4\beta)h_6\left((1+2n)\alpha_1^2 + 8n\alpha_0\alpha_2\right) \\
&\quad + 2\alpha_2^2\left[\lambda^2(2+n)(a-4\beta)h_4 + bn^2\left(3\alpha_1^2 + 2\alpha_0\alpha_2\right)\right] = 0, \\
U^7(\psi) &: \alpha_1\alpha_2\left[\lambda^2(4+7n)(a-4\beta)h_6 + 4bn^2\alpha_2^2\right] = 0, \\
U^8(\psi) &: 4\lambda^2(1+n)(a-4\beta)h_6\alpha_2^2 + bn^2\alpha_2^4 = 0.
\end{aligned} \tag{4.22}$$

We get the following result after solving the resulting system:

$$\begin{aligned}
&\alpha_0 = 0, \quad \alpha_1 = \frac{\lambda\sqrt{1+n}\sqrt{(-a+4\beta)h_4}}{\sqrt{b}n}, \quad h_2 = \frac{n^2\left(ak^2+\gamma+\omega\right)}{\lambda^2(a-4\beta)}, \\
&\alpha_2 = 0, \quad h_0 = 0, \quad h_4 = h_4, \qquad h_6 = 0.
\end{aligned} \tag{4.23}$$

In view of Eqs. (3.14), (4.21) and (4.23) along with the Jacobi elliptic functions in the above table, we obtain the following exact solutions of Eq. (4.18)

1. If $l_0 = 1$, $l_2 = -(1+\rho^2)$, $l_4 = \rho^2$, $0 < \rho < 1$, then $P(\psi) = sn(\psi, \rho)$ or $P(\psi) = cd(\psi, \rho)$, and we have

$$q_{1,1}(x,t)=\left[\frac{\lambda\sqrt{1+n}\sqrt{(-a+4\beta)\,h_4}}{\sqrt{b}n}\left(\frac{sn(\psi,\rho)}{\sqrt{f\,(sn(\psi,\rho))^2+g}}\right)\right]^{\frac{1}{n}}e^{i(-kx+wt+\theta)}, \tag{4.24}$$

or

$$q_{1,2}(x,t)=\left[\frac{\lambda\sqrt{1+n}\sqrt{(-a+4\beta)\,h_4}}{\sqrt{b}n}\left(\frac{cd(\psi,\rho)}{\sqrt{f\,(cd(\psi,\rho))^2+g}}\right)\right]^{\frac{1}{n}}e^{i(-kx+wt+\theta)}, \tag{4.25}$$

such that $0<b,\ \psi=\lambda\,(x-vt)$, and f and g in Eq. (3.16) are given by

$$f=\frac{(1+\rho^2+h_2)h_4}{1-\rho^2+\rho^4-h_2^2},$$
$$g=\frac{-3h_4}{1-\rho^2+\rho^4-h_2^2},$$

under the restriction condition

$$-h_4^2\left(-1-\rho^2-h_2\right)\left(-1+2\rho^2-h_2\right)\left(-2+\rho^2+h_2\right)=0.$$

If $\rho\to 1$, then the dark soliton is obtained

$$q_{1,3}(x,t)=\left[\frac{\lambda\sqrt{1+n}\sqrt{(-a+4\beta)\,h_4}}{\sqrt{b}n}\left(\frac{\tanh(\psi)}{\sqrt{\frac{h_4(3-(2+h_2)\tanh^2(\psi))}{-1+h_2^2}}}\right)\right]^{\frac{1}{n}}e^{i(-kx+wt+\theta)}. \tag{4.26}$$

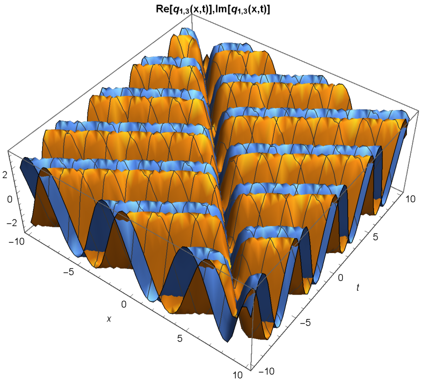
Here in Eq. (4.26), the parameter n dictates the power law nonlinearity. For stability issues, it is necessary to have $0<n<2$, and in particular $n\neq 2$, to avoid self-focusing singularity. Setting $n=1$, results in generelization of kerr law nonlinearity, the dark soliton is now obtained as

$$q_{1,3}(x,t)=\frac{i\sqrt{2}\lambda\sqrt{a-4\beta}e^{i(-kx+\theta+tw)}\tanh(\lambda(2akt+x))}{\sqrt{b}}, \tag{4.27}$$

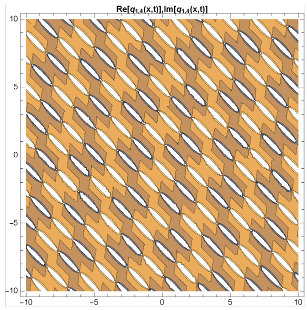
such that

$$h_4^2\left(-2-h_2\right)\left[\left(-1+h_2\right)^2\right]=0.$$

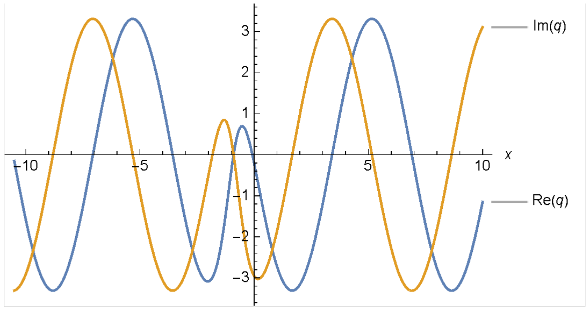
Figure 1 illustrates the dark soliton solution $q_{1,3}(x,t)$ derived in Eq.(4.27) for the limiting case $\rho=1$, where the Jacobi elliptic function reduces to a $\tanh(x)$ profile. The modulus plot (a) and contour plot (b) show the characteristic intensity dip that propagates on a continuous background, confirming the presence of a moving dark soliton. The real and imaginary parts in (c) exhibit a localized π-phase shift, which is a hallmark of dark solitons in defocusing media. In the context of nonlinear fiber optics, such dark solitons arise in fibers exhibiting normal dispersion and self-defocusing nonlinearity, particularly in photonic crystal fibers or fiber Bragg gratings. These solitons are valuable for optical switching, intensity dips for logic gates, and phase-sensitive modulation schemes. Their robustness under power-law nonlinearity suggests potential for advanced pulse shaping and energy channeling in nonlinear waveguides with engineered refractive index profiles. These solutions are physically relevant to nonlinear optical fibers with power-law refractive index profiles, where such localized dips may represent energy voids in high-intensity beam propagation or be employed for pulse modulation and switching.



(A) $|q_{1,3}|$



(B) $|q_{1,3}|$



(C) $|q_{1,3}|$

FIGURE 1. The numerical simulations corresponding to $|q_{1,3}|$ given by Eq. (4.27), for $\rho = 1$; (a) is the 3D graphic, (b) is the 2D-contour graphic while (c) is the 3D graphic for $k = 0.9, \theta = 0.2, \omega = 1.3, a = 0.5, b = 0.7, \lambda = 1.6, \beta = 0.5$.

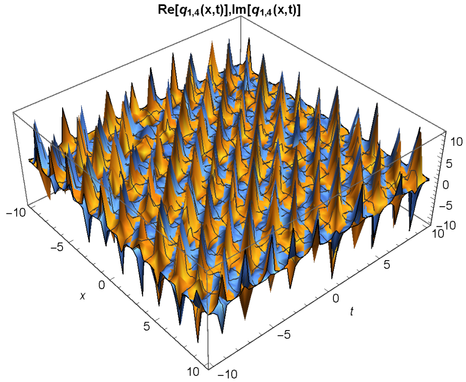
If $\rho \to 0$, then the periodic solution is obtained

$$q_{1,4}\left(x,t\right) = \left[\frac{\lambda\sqrt{1+n}\sqrt{\left(-a+4\beta\right)h_4}}{\sqrt{b}n}\left(\frac{\sin(\psi)}{\sqrt{\frac{h_4(3-(1+h_2)\sin^2(\psi))}{-1+h_2^2}}}\right)\right]^{\frac{1}{n}} e^{i(-kx+wt+\theta)}, \tag{4.28}$$

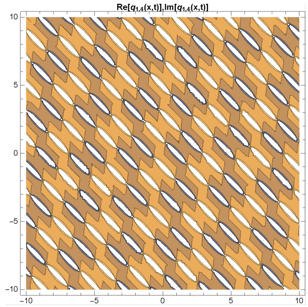
such that

$$h_4^2\left(-1-h_2\right)\left[\left(-2+h_2\right)\left(1+h_2\right)\right] = 0.$$

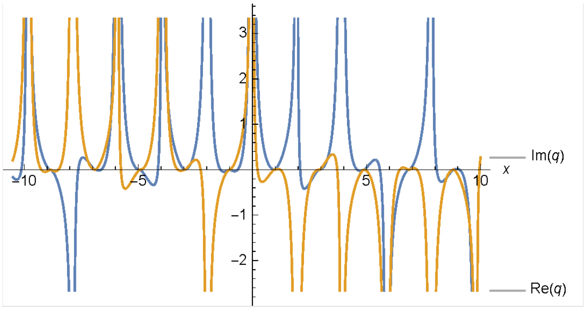
Figure 2 numerically validates the periodic solution $q_{1,4}(x,t)$ Eq. (4.28) for $\rho = 0$, where the Jacobi elliptic function reduces to a sinusoidal profile. The $3D$ plot (a) and contour (b) demonstrate coherent wave propagation with velocity $v = -1$, while the amplitude modulation reflects the nonlinear constraint $h_2 = 2$. Such periodic states are observable in dispersion-managed fibers under pump-driven nonlinearities, offering potential for frequency comb generation or all-optical signal processing. The parameter sensitivity in (c) emphasizes the need for precise control in experimental realizations.



(A) $|q_{1,4}|$



(B) $|q_{1,4}|$



(C) $|q_{1,4}|$

FIGURE 2. The numerical simulations corresponding to $|q_{1,4}|$ given by Eq. (4.28), for $\rho = 0$; (a) is the 3D graphic, (b) is the 2D-contour graphic while (c) is the 3D graphic for $k = 1.8, \theta = 2, \omega = 0.2, v = -1, \gamma = -1.6, \beta = 0.8, a = -0.4, n = 0.7, \lambda = -1.6, h_4 = 2, c = 1.2, \alpha_1 = 1, h_2 = 2$.

2. If $l_0 = 1 - \rho^2$, $l_2 = 2\rho^2 - 1$, $l_4 = -\rho^2$, $0 < \rho < 1$, then $P(\psi) = cn(\psi, \rho)$ therefore

$$q_2(x,t) = \left[\frac{\lambda\sqrt{1+n}\sqrt{(-a+4\beta)\,h_4}}{\sqrt{b}n} \left(\frac{cn(\psi,\rho)}{\sqrt{f\,(cn(\psi,\rho))^2 + g}} \right) \right]^{\frac{1}{n}} e^{i(-kx+wt+\theta)}, \tag{4.29}$$

where f and g are determined by

$$f = -\frac{(-1+2\rho^2-h_2)h_4}{1-\rho^2+\rho^4-h_2^2},$$
$$g = \frac{3\left(-1+\rho^2\right)h_4}{1-\rho^2+\rho^4-h_2^2},$$

under the constraint condition

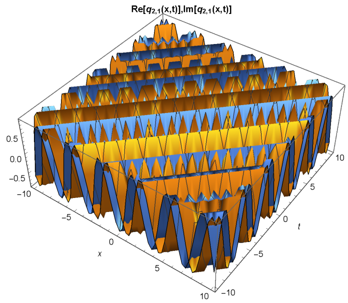
$$h_4^2\left(-1+2\rho^2-h_2\right)\left[\left(-2+\rho^2+h_2\right)\left(1+\rho^2+h_2\right)\right] = 0.$$

If $\rho \to 1$, then the bright soliton is retrieved

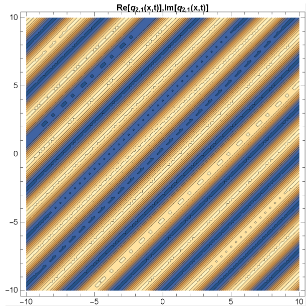
$$q_{2,1}(x,t) = \left[\frac{\lambda\sqrt{1+n}\sqrt{(-a+4\beta)\,h_4}}{\sqrt{bn}}\left(\frac{sech(\psi)}{\sqrt{\frac{-h_4 sech^2(\psi)}{1+h_2}}}\right)\right]^{\frac{1}{n}} e^{i(-kx+wt+\theta)} \tag{4.30}$$

provided that

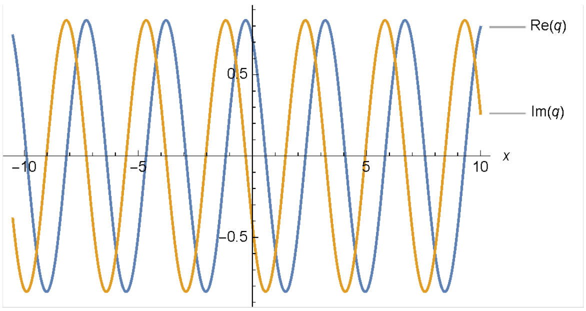
$$h_4^2\,(1-h_2)\left[h_2^2+h_2-2\right]=0.$$



(A) $|q_{2,1}|$



(B) $|q_{2,1}|$



(C) $|q_{2,1}|$

FIGURE 3. The numerical simulations corresponding to $|q_{2,1}|$ given by Eq. (4.30), for $\rho = 1$; (a) is the 3D graphic , (b) is the 2D-contour graphic while (c) is the 3D graphic for $k = 1.8, \theta = 5, \omega = 1.8, v = 1, \gamma = 1, \beta = 0.8, a = 1, n = 1.5, \lambda = 1.6, h_4 = -2, c = 1.2, \alpha_1 = 1.078, h_2 = -2$.

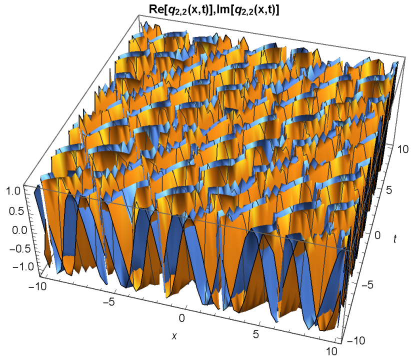
Figure 3 numerically validates the bright soliton solution $q_{2,1}$ Eq. (4.30) for $\rho = 1$, where the Jacobi elliptic function reduces to $\operatorname{sech}(\psi)\operatorname{sech}(\psi)$. The 3D plot (a) and contour (b) demonstrate the solitons stable propagation, with the constraint $h_2 = -2$ ensuring a physical amplitude. Panel (c) reveals how parameter choices (e.g., $h_4 = -2$) tune the pulse shape critical for designing soliton-based fiber-optic systems. Such solutions are experimentally achievable in fibers with anomalous dispersion ($a < 4$) and power levels balancing nonlinearity and dispersion.

If $\rho \to 0$, then the periodic solution is obtained

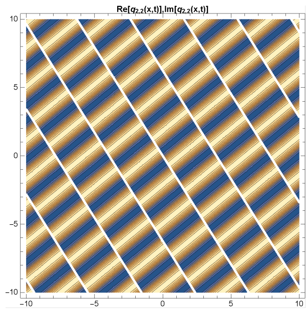
$$q_{2,2}(x,t) = \left[\frac{\lambda\sqrt{1+n}\sqrt{(-a+4\beta)\,h_4}}{\sqrt{bn}}\left(\frac{\cos(\psi)}{\sqrt{\frac{-h_4(-3+(1+h_2)\cos^2(\psi))}{-1+h_2^2}}}\right)\right]^{\frac{1}{n}} e^{i(-kx+wt+\theta)}, \tag{4.31}$$

such that

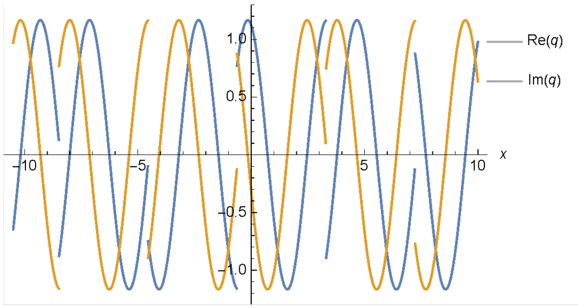
$$h_4^2\,(-1-h_2)\,[(-2+h_2)\,(1+h_2)]=0.$$



(A) $|q_{2,2}|$



(B) $|q_{2,2}|$



(C) $|q_{2,2}|$

FIGURE 4. The numerical simulations corresponding to $|q_{2,2}|$ given by Eq. (4.31), for $\rho = 0$; (a) is the 3D graphic, (b) is the 2D-contour graphic while (c) is the 3D graphic for $k = 1.8, \theta = 5, \omega = 2.2, v = -0.63, \gamma = 1, \beta = 0.8, a = 1, n = 1.32, \lambda = 0.8, h_4 = 2, c = 1.2, \alpha_1 = 1, h_2 = 2$.

Figure 4 numerically validates the periodic solution $q_{2,2}$ Eq. (4.31) for $\rho = 0$, where the profile reduces to a $\cos(\psi)$ wave. The 3D plot (a) and contour (b) demonstrate coherent propagation with velocity $v = -0.63$, while the amplitude modulation in (c) reflects the nonlinear constraint $h_2 = 2$. Such solutions model stimulated Brillouin scattering in fibers or optical lattice vibrations in waveguide arrays, with nodes/antinodes tunable via h_4 and λ.

3. If $l_0 = \rho^2 - 1, l_2 = 2 - \rho^2, l_4 = -1, 0 < \rho < 1$, then $P(\psi) = dn(\psi, \rho)$ which gives

$$q_3\,(x,t) = \left[\frac{\lambda\sqrt{1+n}\sqrt{(-a+4\beta)\,h_4}}{\sqrt{b}n}\left(\frac{dn(\psi,\rho)}{\sqrt{f\,(dn(\psi,\rho))^2+g}}\right)\right]^{\frac{1}{n}} e^{i(-kx+wt+\theta)}, \tag{4.32}$$

where f and g are determined by

$$f = \frac{(-2+\rho^2+h_2)h_4}{1-\rho^2+\rho^4-h_2^2},$$
$$g = \frac{-3\left(-1+\rho^2\right)h_4}{1-\rho^2+\rho^4-h_2^2},$$

under the restriction condition

$$h_4^2\left(2-\rho^2-h_2\right)\left[-\left(-1+2\rho^2+h_2\right)\left(1+\rho^2+h_2\right)\right]=0.$$

If $\rho \to 1$, then the singular soliton solution is obtained

$$q_{3,1}(x,t) = \left[\frac{\lambda\sqrt{1+n}\sqrt{(-a+4\beta)\,h_4}}{\sqrt{b}n}\left(\frac{sech(\psi)}{\sqrt{\frac{-h_4 sech^2(\psi)}{1+h_2}}}\right)\right]^{\frac{1}{n}} e^{i(-kx+wt+\theta)}, \tag{4.33}$$

provided that

$$h_4^2\left(1-h_2\right)\left[-2+h_2+h_2^2\right]=0.$$

If $\rho \to 0$, then the rational solution is obtained

$$q_{3,2}(x,t) = \left[\frac{\lambda\sqrt{1+n}\sqrt{(-a+4\beta)\,h_4}}{\sqrt{b}n}\left(\frac{1}{\sqrt{\frac{h_4}{1-h_2}}}\right)\right]^{\frac{1}{n}} e^{i(-kx+wt+\theta)}, \tag{4.34}$$

such that

$$h_4^2\left(2-h_2\right)\left[\left(1+h_2\right)^2\right]=0.$$

4. If $l_0=\rho^2$, $l_2=-\left(1+\rho^2\right)$, $l_4=1$, $0<\rho<1$, then $P(\psi)=ns(\psi,\rho)$ or $P(\psi)=dc(\psi,\rho)$ then

$$q_{4,1}(x,t) = \left[\frac{\lambda\sqrt{1+n}\sqrt{(-a+4\beta)\,h_4}}{\sqrt{b}n}\left(\frac{ns(\psi,\rho)}{\sqrt{f\left(ns(\psi,\rho)\right)^2+g}}\right)\right]^{\frac{1}{n}} e^{i(-kx+wt+\theta)}, \tag{4.35}$$

or

$$q_{4,2}(x,t) = \left[\frac{\lambda\sqrt{1+n}\sqrt{(-a+4\beta)\,h_4}}{\sqrt{b}n}\left(\frac{dc(\psi,\rho)}{\sqrt{f\left(dc(\psi,\rho)\right)^2+g}}\right)\right]^{\frac{1}{n}} e^{i(-kx+wt+\theta)}, \tag{4.36}$$

where f and g are given by

$$f = \frac{(1+\rho^2+h_2)h_4}{1-\rho^2+\rho^4-h_2^2},$$
$$g = \frac{-3\rho^2 h_4}{1-\rho^2+\rho^4-h_2^2},$$

under the constraint condition

$$h_4^2\left(-1-\rho^2-h_2\right)\left[-\left(-1+2\rho^2-h_2\right)\left(-2+\rho^2+h_2\right)\right]=0.$$

If $\rho \to 1$, then the dark singular solution is obtained

$$q_{4,3}(x,t) = \left[\frac{\lambda\sqrt{1+n}\sqrt{(-a+4\beta)\,h_4}}{\sqrt{b}n}\left(\frac{\coth(\psi)}{\sqrt{\frac{(-1+h_2+(2+h_2)\operatorname{csch}^2(\psi))h_4}{1-h_2^2}}}\right)\right]^{\frac{1}{n}} e^{i(-kx+wt+\theta)} \tag{4.37}$$

such that

$$h_4^2\left(-2-h_2\right)\left[\left(-1+h_2\right)^2\right]=0.$$

If $\rho\to 0$, then the periodic solution is obtained

$$q_{4,4}\left(x,t\right)=\left[\frac{\lambda\sqrt{1+n}\sqrt{\left(-a+4\beta\right)h_4}}{\sqrt{b}n}\left(\frac{\csc(\psi)}{\sqrt{\frac{-\csc^2(\psi)h_4}{-1+h_2}}}\right)\right]^{\frac{1}{n}}e^{i(-kx+wt+\theta)}, \tag{4.38}$$

such that

$$h_4^2\left(-1-h_2\right)\left[\left(-2+h_2\right)\left(1+h_2\right)\right]=0.$$

5. If $l_0=-\rho^2$, $l_2=2\rho^2-1$, $l_4=1-\rho^2$, $0<\rho<1$, then $P(\psi)=nc(\psi,\rho)$ and we have

$$q_5\left(x,t\right)=\left[\frac{\lambda\sqrt{1+n}\sqrt{\left(-a+4\beta\right)h_4}}{\sqrt{b}n}\left(\frac{nc(\psi,\rho)}{\sqrt{f\left(nc(\psi,\rho)\right)^2+g}}\right)\right]^{\frac{1}{n}}e^{i(-kx+wt+\theta)}, \tag{4.39}$$

where f and g are given by

$$f=\frac{-(-1+2\rho^2-h_2)h_4}{1-\rho^2+\rho^4-h_2^2},$$
$$g=\frac{3\rho^2h_4}{1-\rho^2+\rho^4-h_2^2},$$

under the constraint condition

$$h_4^2\left(-1+2\rho^2-h_2\right)\left[\left(-2+\rho^2+h_2\right)\left(1+\rho^2+h_2\right)\right]=0.$$

If $\rho\to 1$, then the solitary wave solution is obtained

$$q_{5,1}\left(x,t\right)=\left[\frac{\lambda\sqrt{1+n}\sqrt{\left(-a+4\beta\right)h_4}}{\sqrt{b}n}\left(\frac{\cosh(\psi)}{\sqrt{\frac{(-3+(1-h_2)\cosh^2(\psi))h_4}{-1+h_2^2}}}\right)\right]^{\frac{1}{n}}e^{i(-kx+wt+\theta)}, \tag{4.40}$$

such that

$$h_4^2\left(1-h_2\right)\left[-2+h_2+h_2^2\right]=0.$$

If $\rho\to 0$, then the periodic solution is obtained

$$q_{5,2}\left(x,t\right)=\left[\frac{\lambda\sqrt{1+n}\sqrt{\left(-a+4\beta\right)h_4}}{\sqrt{b}n}\left(\frac{\sec(\psi)}{\sqrt{\frac{-h_4\sec^2(\psi)}{-1+h_2}}}\right)\right]^{\frac{1}{n}}e^{i(-kx+wt+\theta)}, \tag{4.41}$$

such that

$$h_4^2\left(-1-h_2\right)\left[\left(-2+h_2\right)\left(1+h_2\right)\right]=0.$$

6. If $l_0=-1$, $l_2=2-\rho^2$, $l_4=-\left(1-\rho^2\right)$, $0<\rho<1$, then $P(\psi)=nd(\psi,\rho)$ and we have

$$q_6\left(x,t\right)=\left[\frac{\lambda\sqrt{1+n}\sqrt{\left(-a+4\beta\right)h_4}}{\sqrt{b}n}\left(\frac{nd(\psi,\rho)}{\sqrt{f\left(nd(\psi,\rho)\right)^2+g}}\right)\right]^{\frac{1}{n}}e^{i(-kx+wt+\theta)}, \tag{4.42}$$

where f and g are given by

$$f = \frac{(-2+\rho^2+h_2)h_4}{1-\rho^2+\rho^4-h_2^2},$$
$$g = \frac{3h_4}{1-\rho^2+\rho^4-h_2^2},$$

under the constraint condition

$$h_4^2\left(2-\rho^2-h_2\right)\left[-\left(-1+2\rho^2-h_2\right)\left(1+\rho^2+h_2\right)\right]=0.$$

7. If $l_0 = 1$, $l_2 = 2-\rho^2$, $l_4 = 1-\rho^2$, $0<\rho<1$, then $P(\psi) = sc(\psi,\rho)$, and we have

$$q_7\left(x,t\right) = \left[\frac{\lambda\sqrt{1+n}\sqrt{\left(-a+4\beta\right)h_4}}{\sqrt{b}n}\left(\frac{sc(\psi,\rho)}{\sqrt{f\left(sc(\psi,\rho)\right)^2+g}}\right)\right]^{\frac{1}{n}} e^{i(-kx+wt+\theta)}, \tag{4.43}$$

where f and g are given by

$$f = \frac{(-2+\rho^2+h_2)h_4}{1-\rho^2+\rho^4-h_2^2},$$
$$g = \frac{-3h_4}{1-\rho^2+\rho^4-h_2^2},$$

under the constraint condition

$$h_4^2\left(2-\rho^2-h_2\right)\left[-\left(-1+2\rho^2-h_2\right)\left(1+\rho^2+h_2\right)\right]=0.$$

If $\rho\to 1$, then the singular soliton solution is obtained

$$q_{7,1}\left(x,t\right) = \left[\frac{\lambda\sqrt{1+n}\sqrt{\left(-a+4\beta\right)h_4}}{\sqrt{b}n}\left(\frac{\sinh(\psi)}{\sqrt{\frac{(3+(1-h_2)\sinh^2(\psi))h_4}{-1+h_2^2}}}\right)\right]^{\frac{1}{n}} e^{i(-kx+wt+\theta)}, \tag{4.44}$$

such that

$$h_4^2\left(1-h_2\right)\left[-2+h_2+h_2^2\right]=0.$$

If $\rho\to 0$, then the periodic wave solution is obtained

$$q_{7,2}\left(x,t\right) = \left[\frac{\lambda\sqrt{1+n}\sqrt{\left(-a+4\beta\right)h_4}}{\sqrt{b}n}\left(\frac{\tan(\psi)}{\sqrt{\frac{(3-(-2+h_2)\tan^2(\psi))h_4}{-1+h_2^2}}}\right)\right]^{\frac{1}{n}} e^{i(-kx+wt+\theta)}, \tag{4.45}$$

such that

$$h_4^2\left(2-h_2\right)\left[\left(1+h_2\right)^2\right]=0.$$

8. If $l_0 = 1$, $l_2 = 2\rho^2-1$, $l_4 = -\rho^2\left(1-\rho^2\right)$, $0<\rho<1$, then $P(\psi) = sd(\psi,\rho)$ and we have

$$q_8\left(x,t\right) = \left[\frac{\lambda\sqrt{1+n}\sqrt{\left(-a+4\beta\right)h_4}}{\sqrt{b}n}\left(\frac{sd(\psi,\rho)}{\sqrt{f\left(sd(\psi,\rho)\right)^2+g}}\right)\right]^{\frac{1}{n}} e^{i(-kx+wt+\theta)}, \tag{4.46}$$

where f and g are given by

$$f = \frac{(-1+2\rho^2-h_2)h_4}{1-\rho^2+\rho^4-h_2^2},$$
$$g = \frac{-3h_4}{1-\rho^2+\rho^4-h_2^2},$$

under the constraint condition

$$h_4^2\left(-1+2\rho^2-h_2\right)\left[\left(-2+\rho^2+h_2\right)\left(1+\rho^2+h_2\right)\right]=0.$$

9. If $l_0 = 1-\rho^2$, $l_2 = 2-\rho^2$, $l_4 = 1$, $0<\rho<1$, then $P(\psi) = cs(\psi,\rho)$ and we have

$$q_9(x,t) = \left[\frac{\lambda\sqrt{1+n}\sqrt{(-a+4\beta)h_4}}{\sqrt{b}n}\left(\frac{cs(\psi,\rho)}{\sqrt{f\left(cs(\psi,\rho)\right)^2+g}}\right)\right]^{\frac{1}{n}} e^{i(-kx+wt+\theta)}, \tag{4.47}$$

where f and g are given by

$$f = \frac{(-2+\rho^2+h_2)h_4}{1-\rho^2+\rho^4-h_2^2},$$
$$g = \frac{3(-1+\rho^2)h_4}{1-\rho^2+\rho^4-h_2^2},$$

under the constraint condition

$$h_4^2\left(2-\rho^2-h_2\right)\left[-\left(-1+2\rho^2-h_2\right)\left(1+\rho^2+h_2\right)\right]=0.$$

If $\rho\to 1$, then the singular soliton solution is obtained

$$q_{9,1}(x,t) = \left[\frac{\lambda\sqrt{1+n}\sqrt{(-a+4\beta)h_4}}{\sqrt{b}n}\left(\frac{\operatorname{csch}(\psi)}{\sqrt{\frac{-\operatorname{csch}^2(\psi)h_4}{1+h_2}}}\right)\right]^{\frac{1}{n}} e^{i(-kx+wt+\theta)}, \tag{4.48}$$

such that

$$h_4^2\left(1-h_2\right)\left[-2+h_2+h_2^2\right]=0.$$

If $\rho\to 0$, then the periodic wave solution is obtained

$$q_{9,2}(x,t) = \left[\frac{\lambda\sqrt{1+n}\sqrt{(-a+4\beta)h_4}}{\sqrt{b}n}\left(\frac{\cot(\psi)}{\sqrt{\frac{(3+(2-h_2)\cot^2(\psi))h_4}{-1+h_2^2}}}\right)\right]^{\frac{1}{n}} e^{i(-kx+wt+\theta)}, \tag{4.49}$$

such that

$$h_4^2\left(2-h_2\right)\left[\left(1+h_2\right)^2\right]=0.$$

10. If $l_0 = -\rho^2\left(1-\rho^2\right)$, $l_2 = 2\rho^2-1$, $l_4 = 1$, $0<\rho<1$, then $P(\psi) = ds(\psi,\rho)$ and we have

$$q_{10}(x,t) = \left[\frac{\lambda\sqrt{1+n}\sqrt{(-a+4\beta)h_4}}{\sqrt{b}n}\left(\frac{ds(\psi,\rho)}{\sqrt{f\left(ds(\psi,\rho)\right)^2+g}}\right)\right]^{\frac{1}{n}} e^{i(-kx+wt+\theta)}, \tag{4.50}$$

where f and g are given by

$$f = \frac{-(-1+2\rho^2-h_2)h_4}{1-\rho^2+\rho^4-h_2^2},$$
$$g = \frac{-3\rho^2(-1+\rho^2)h_4}{1-\rho^2+\rho^4-h_2^2},$$

under the constraint condition

$$h_4^2\left(-1+2\rho^2-h_2\right)\left[\left(-2+\rho^2+h_2\right)\left(1+\rho^2+h_2\right)\right]=0.$$

11. If $l_0 = \frac{1-\rho^2}{4}$, $l_2 = \frac{1+\rho^2}{2}$, $l_4 = \frac{1-\rho^2}{4}$, $0<\rho<1$, then $P(\psi) = nc(\psi,\rho) \pm sc(\psi,\rho)$ or $P(\psi) = \frac{cn(\psi,\rho)}{1\pm sn(\psi,\rho)}$ and we have

(4.51)
$$q_{11,1}(x,t) = \left[\frac{\lambda\sqrt{1+n}\sqrt{(-a+4\beta)h_4}}{\sqrt{b}n}\left(\frac{nc(\psi,\rho)\pm sc(\psi,\rho)}{\sqrt{f\left(nc(\psi,\rho)\pm sc(\psi,\rho)\right)^2+g}}\right)\right]^{\frac{1}{n}} e^{i(-kx+wt+\theta)},$$

or

$$q_{11,2}(x,t) = \left[\frac{\lambda\sqrt{1+n}\sqrt{(-a+4\beta)h_4}}{\sqrt{b}n}\left(\frac{\frac{cn(\psi,\rho)}{1\pm sn(\psi,\rho)}}{\sqrt{f\left(\frac{cn(\psi,\rho)}{1\pm sn(\psi,\rho)}\right)^2+g}}\right)\right]^{\frac{1}{n}} e^{i(-kx+wt+\theta)}, \tag{4.52}$$

where f and g are given by

$$f = \frac{-8(1+\rho^2-2h_2)h_4}{1+14\rho^2+\rho^4-16h_2^2},$$
$$g = \frac{12(-1+\rho^2)h_4}{1+14\rho^2+\rho^4-16h_2^2},$$

under the constraint condition

$$h_4^2\left(\frac{1}{2}\left(1+\rho^2-2h_2\right)\right)\left[\frac{1}{16}\left(1+(-6+\rho)\rho+4h_2\right)\left(1+\rho(6+\rho)+4h_2\right)\right]=0.$$

If $\rho\to 1$, then the combined singular soliton solution is obtained

$$q_{11,3}(x,t) = \left[\frac{\lambda\sqrt{1+n}\sqrt{(-a+4\beta)h_4}}{\sqrt{b}n}\left(\frac{\sinh(\psi)+\cosh(\psi)}{\sqrt{\frac{-h_4(\sinh(\psi)+\cosh(\psi))^2}{1+h_2}}}\right)\right]^{\frac{1}{n}} e^{i(-kx+wt+\theta)}, \tag{4.53}$$

or

$$q_{11,4}(x,t) = \left[\frac{\lambda\sqrt{1+n}\sqrt{(-a+4\beta)h_4}}{\sqrt{b}n}\left(\frac{\cosh(\psi)-\sinh(\psi)}{\sqrt{\frac{-h_4(\cosh(\psi)-\sinh(\psi))^2}{1+h_2}}}\right)\right]^{\frac{1}{n}} e^{i(-kx+wt+\theta)}, \tag{4.54}$$

such that

$$h_4^2\left(1-h_2\right)\left[-2+h_2+h_2^2\right]=0.$$

If $\rho \to 0$, then the combined periodic wave solutions
(4.55)

$$q_{11,5}(x,t) = \frac{1}{\sqrt{2}}\left[\frac{\lambda\sqrt{1+n}\sqrt{(-a+4\beta)h_4}}{\sqrt{b}n}\left(\frac{\sec(\psi)+\tan(\psi)}{\sqrt{\frac{h_4(-5+4h_2+(1+4h_2)\sin(\psi))}{(-1+\sin(\psi))\left(-1+16h_2^2\right)}}}\right)\right]^{\frac{1}{n}} e^{i(-kx+wt+\theta)},$$

or

$$q_{11,6}(x,t) = \left[\frac{\lambda\sqrt{1+n}\sqrt{(-a+4\beta)h_4}}{2\sqrt{b}n}\left(\frac{\frac{\cos(\psi)}{1+\sin(\psi)}}{\sqrt{\frac{h_4(5-4h_2+(1+4h_2)\sin(\psi))}{(1+\sin(\psi))\left(-1+16h_2^2\right)}}}\right)\right]^{\frac{1}{n}} e^{i(-kx+wt+\theta)}, \tag{4.56}$$

are obtained, such that

$$h_4^2\left(\frac{1}{2}-h_2\right)\left[\frac{1}{16}(1+4h_2)^2\right]=0.$$

12. If $l_0 = \frac{-\left(1-\rho^2\right)^2}{4}$, $l_2 = \frac{1+\rho^2}{2}$, $l_4 = \frac{-1}{4}$, $0<\rho<1$, then $P(\psi) = \rho cn(\psi,\rho) \pm dn(\psi,\rho)$ and we have

(4.57)

$$q_{12}(x,t) = \left[\frac{\lambda\sqrt{1+n}\sqrt{(-a+4\beta)h_4}}{\sqrt{b}n}\left(\frac{\rho cn(\psi,\rho)\pm dn(\psi,\rho)}{\sqrt{f\left(\rho cn(\psi,\rho)\pm dn(\psi,\rho)\right)^2+g}}\right)\right]^{\frac{1}{n}} e^{i(-kx+wt+\theta)},$$

where f and g are given by

$$f = \frac{-8(1+\rho^2-2h_2)h_4}{1+14\rho^2+\rho^4-16h_2^2},$$
$$g = \frac{12(-1+\rho^2)^2h_4}{1+14\rho^2+\rho^4-16h_2^2},$$

under the constraint condition

$$h_4^2\left(\frac{1}{2}\left(1+\rho^2-2h_2\right)\right)\left[\frac{1}{16}\left(1+(-6+\rho)\rho+4h_2\right)\left(1+\rho(6+\rho)+4h_2\right)\right]=0.$$

If $\rho \to 1$, then the singular soliton solution is obtained

$$q_{12,1}(x,t) = \left[\frac{\lambda\sqrt{1+n}\sqrt{(-a+4\beta)h_4}}{\sqrt{b}n}\left(\frac{sech(\psi)}{\sqrt{\frac{-h_4 sech^2(\psi)}{1+h_2}}}\right)\right]^{\frac{1}{n}} e^{i(-kx+wt+\theta)} \tag{4.58}$$

such that

$$h_4^2(1-h_2)\left[-2+h_2+h_2^2\right]=0.$$

If $\rho \to 0$, then the periodic wave solution is obtained

$$q_{12,2}(x,t) = \left[\frac{\lambda\sqrt{1+n}\sqrt{(-a+4\beta)h_4}}{2\sqrt{b}n}\left(\frac{\cos(\psi)}{\sqrt{\frac{h_4(-2+\cos(2\psi)-4h_2\cos^2(\psi))}{-1+16h_2^2}}}\right)\right]^{\frac{1}{n}} e^{i(-kx+wt+\theta)}, \tag{4.59}$$

such that

$$h_4^2\left(\frac{1}{2}-h_2\right)\left[\frac{1}{16}\left(1+4h_2\right)^2\right]=0.$$

13. If $l_0=\frac{1}{4}, l_2=\frac{1-2\rho^2}{2}, l_4=\frac{1}{4}, 0<\rho<1$, then $P(\psi)=\frac{sn(\psi,\rho)}{1\pm cn(\psi,\rho)}$ and we have

$$q_{13}\left(x,t\right)=\left[\frac{\lambda\sqrt{1+n}\sqrt{\left(-a+4\beta\right)h_4}}{\sqrt{b}n}\left(\frac{\frac{sn(\psi,\rho)}{1\pm cn(\psi,\rho)}}{\sqrt{f\left(\frac{sn(\psi,\rho)}{1\pm cn(\psi,\rho)}\right)^2+g}}\right)\right]^{\frac{1}{n}}e^{i(-kx+wt+\theta)}, \tag{4.60}$$

where f and g are given by

$$f=\frac{8(-1+2\rho^2+2h_2)h_4}{1-16\rho^2+16\rho^4-16h_2^2},$$
$$g=\frac{-12h_4}{1-16\rho^2+16\rho^4-16h_2^2},$$

under the constraint condition

$$h_4^2\left(\frac{1}{2}-\rho^2-h_2\right)\left[\frac{1}{16}+2\rho^2-2\rho^4+\left(\frac{1}{2}-\rho^2\right)h_2+h_2^2\right]=0.$$

If $\rho\to 1$, then the combined solitary solution is obtained

$$q_{13,1}\left(x,t\right)=\left[\frac{\lambda\sqrt{1+n}\sqrt{\left(-a+4\beta\right)h_4}}{2\sqrt{b}n}\left(\frac{\tanh(\frac{\psi}{2})}{\sqrt{\frac{h_4(5+(1-4h_2)\cosh(\psi)+4h_2)}{(1+\cosh(\psi))(-1+16h_2^2)}}}\right)\right]^{\frac{1}{n}}e^{i(-kx+wt+\theta)}, \tag{4.61}$$

such that

$$h_4^2\left(\frac{-1}{2}-h_2\right)\left[\frac{1}{16}\left(1-4h_2\right)^2\right]=0.$$

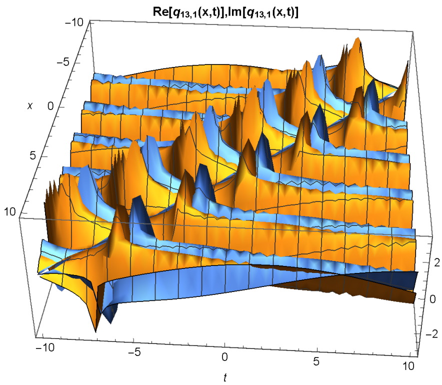
If $\rho\to 0$, then the combined periodic wave solution is obtained

$$q_{13,2}\left(x,t\right)=\left[\frac{\lambda\sqrt{1+n}\sqrt{\left(-a+4\beta\right)h_4}}{2\sqrt{b}n}\left(\frac{\frac{\sin(\psi)}{1+\cos(\psi)}}{\sqrt{\frac{h_4\left(3+2(1-2h_2)\left(\frac{\sin(\psi)}{1+\cos(\psi)}\right)^2\right)}{(-1+16h_2^2)}}}\right)\right]^{\frac{1}{n}}e^{i(-kx+wt+\theta)}, \tag{4.62}$$

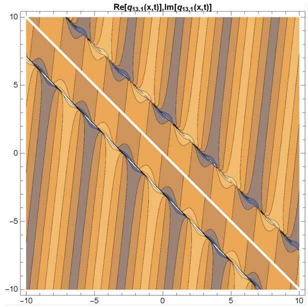
such that

$$h_4^2\left(\frac{1}{2}-h_2\right)\left[\frac{1}{16}\left(1+4h_2\right)^2\right]=0.$$

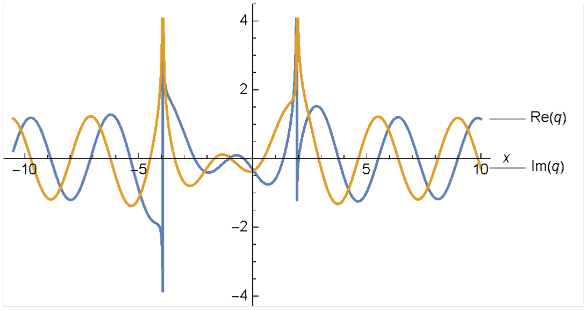
Figures 5 and 6 showcase the remarkable duality of nonlinear wave solutions in the φ^6-CGLE system, spanning from singular solitons to periodic waves. Figure 5 captures the singular soliton $q_{13,1}$ ($\rho\to 1$), characterized by a sharp, kink-like intensity dip (panel a) and stable propagation (panel b), arising from the interplay of anomalous dispersion ($a=1$) and focusing nonlinearity ($h_4=-0.82$). This solution models Optical rogue waves or topological defects, with its phase discontinuity offering potential applications in shock wave generation and singular optics. In contrast, Figure 6 displays a periodic solution $q_{13,2}$ ($m\to 0$), where trigonometric modulation (panel a) and coherent wavefronts (panel b) reflect parametric wave mixing or Brillouin scattering in dispersion-managed systems. The transition from singular Figure 5 to periodic Figure 6 states controlled by the elliptic modulus ρ highlights the system's



(A) $|q_{13,1}|$



(B) $|q_{13,1}|$



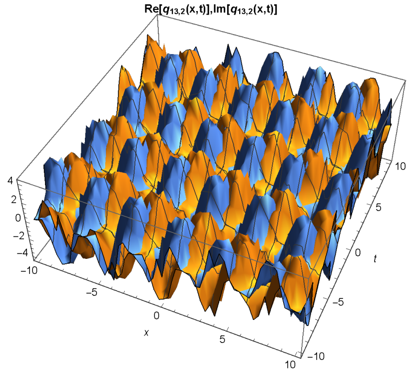
(C) $|q_{13,1}|$

FIGURE 5. The numerical simulations corresponding to $|q_{13,1}|$ given by Eq. (4.61), for $\rho = 1$; (a) is the 3D graphic, (b) is the 2D-contour graphic while (c) is the 3D graphic for $k = 1.8, \theta = 5, \omega = 0.2, v = -1, \gamma = 1, \beta = 0.8, a = 1, n = 1.3, \lambda = 0.6, h_4 = -0.82, c = 1.2, \alpha_1 = 1, h_2 = 1$.

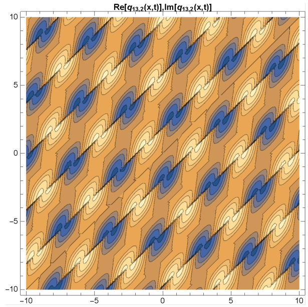
versatility, with h_2 and h_4 tuning amplitude and stability. Together, these figures demonstrate the φ^6-CGLE capacity to support diverse wave phenomena, from localized singularities to extended periodic patterns, paving the way for advanced optical technologies in frequency comb generation, soliton communication, and nonlinear signal processing. This concludes our analysis, underscoring the rich nonlinear dynamics achievable through parameter engineering in higher-order wave systems.

14. If $l_0 = \frac{1}{4}, l_2 = \frac{1+\rho^2}{2}, l_4 = \frac{\left(1-\rho^2\right)^2}{4}, 0 < \rho < 1$, then $P(\psi) = \frac{sn(\psi,\rho)}{cn(\psi,\rho) \pm dn(\psi,\rho)}$ and we have

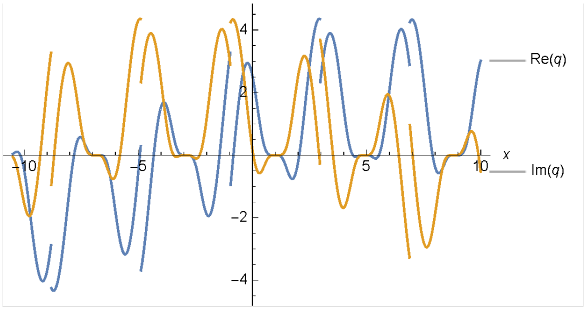
$$q_{14}(x,t) = \left[\frac{\lambda\sqrt{1+n}\sqrt{(-a+4\beta)\,h_4}}{\sqrt{b}n}\left(\frac{\frac{sn(\psi,\rho)}{cn(\psi,\rho)\pm dn(\psi,\rho)}}{\sqrt{f\left(\frac{sn(\psi,\rho)}{cn(\psi,\rho)\pm dn(\psi,\rho)}\right)^2+g}}\right)\right]^{\frac{1}{n}} e^{i(-kx+wt+\theta)}, \tag{4.63}$$



(A) $|q_{13,2}|$



(B) $|q_{13,2}|$



(C) $|q_{13,2}|$

FIGURE 6. The numerical simulations corresponding to $|q_{13,2}|$ given by Eq. (4.62), for $\rho = 0$; (a) is the 3D graphic, (b) is the 2D-contour graphic while (c) is the 3D graphic for $k = 1.8, \theta = 5, \omega = 0.26, v = 1, \gamma = 1, \beta = 0.8, a = 1, n = 0.23, \lambda = 1.6, h_4 = 0.8, c = 1.2, \alpha_1 = 1, h_2 = -2$.

where f and g are given by

$$f = \frac{-8(1+\rho^2-2h_2)h_4}{1+14\rho^2+\rho^4-16h_2^2},$$
$$g = \frac{-12h_4}{1+14\rho^2+\rho^4-16h_2^2},$$

under the constraint condition

$$h_4^2\left(\frac{1}{2}\left(1+\rho^2-2h_2\right)\right)\left[\frac{1}{16}\left(1+\left(-6+\rho\right)\rho+4h_2\right)\left(1+\rho\left(6+\rho\right)+4h_2\right)\right] = 0.$$

If $\rho \to 1$, then the singular soliton solution is obtained

$$q_{14,1}\left(x,t\right) = \left[\frac{\lambda\sqrt{1+n}\sqrt{\left(-a+4\beta\right)h_4}}{\sqrt{b}n}\left(\frac{\sinh(\psi)}{\sqrt{\frac{h_4(3+(1-h_2)\sinh^2(\psi))}{-1+h_2^2}}}\right)\right]^{\frac{1}{n}} e^{i(-kx+wt+\theta)}, \tag{4.64}$$

such that

$$h_4^2\left(1-h_2\right)\left[-2+h_2+h_2^2\right] = 0.$$

If $\rho \to 0$, then the periodic wave is obtained

$$(4.65)\quad q_{14,2}(x,t) = \left[\frac{\lambda\sqrt{1+n}\sqrt{(-a+4\beta)\, h_4}}{2\sqrt{b}n} \left(\frac{\frac{\sin(\psi)}{1+\cos(\psi)}}{\sqrt{\frac{h_4\left(3+2(1-2h_2)\left(\frac{\sin(\psi)}{1+\cos(\psi)}\right)^2\right)}{-1+16h_2^2}}} \right) \right]^{\frac{1}{n}} e^{i(-kx+wt+\theta)},$$

such that

$$h_4^2 \left(\frac{1}{2} - h_2\right) \left[\frac{1}{16}\,(1+4h_2)^2\right] = 0.$$

5. Remarks

This study examines the Complex GinzburgLandau Equation (CGLE) for soliton propagation in nonlinear optics, accounting for the presence of a detuning factor. By applying the φ^6-model expansion method, explicit solutions for bright, dark, periodic, and darkbright solitons are obtained, along with singular soliton solutions. The analysis is conducted within the framework of power-law nonlinear fibers. The findings are expected to enhance understanding of the nonlinear dynamical properties of the CGLE. The proposed method offers an efficient and practical approach for deriving exact solutions to a broad class of nonlinear fractional partial differential equations.

Figures 1–3 illustrate the temporal behavior of dark, bright, darkbright, periodic, and combined periodic wave solutions, which are relevant to the transmission of energy between spatial locations. Additionally, the study explores the physical interpretation of the parameters involved in the classical wave transformation, described by Eqs. (2.1) and (2.2). The solutions to Eqs. (4.26), (4.28), (4.30), (4.31), (4.61), and (4.62) describe traveling waves that incorporate various mathematical constants, reflecting the internal dynamics of the wave under different parameter values. It is observed that variations in these parameters lead to notable changes in the traveling wave behavior. In particular, the discussion highlights how changes in the soliton frequency k influence one of the key internal dynamics of the traveling wave.

6. Conclusion

In this study, we have investigated the propagation of optical solitons in nonlinear fibers governed by the Complex GinzburgLandau Equation (CGLE) with power-law nonlinearity, incorporating the physical effect of detuning. By employing the recently developed φ^6model expansion method, we have successfully derived a rich variety of exact traveling wave solutions, including bright solitons, dark solitons, singular solitons, periodic waves, and several hybrid structures such as darkbright and combined singular forms. The method demonstrated remarkable efficiency and flexibility in handling the nonlinear structure of the CGLE, yielding solutions expressed in terms of Jacobi elliptic functions, hyperbolic, trigonometric, rational, and mixed functional forms. In the limiting cases where the modulus $\rho \to 1$ or $\rho \to 0$, these general solutions reduce to well-known localized solitons or periodic wave patterns, confirming their physical consistency and dynamical relevance. In particular, the emergence of both bright and dark solitons under appropriate parametric conditions highlights the model's ability to describe diverse nonlinear wave phenomena in optical media with power-law dependence.

From a physical standpoint, the derived solutions contribute meaningfully to the understanding of non-linear wave dynamics in optical fibers with power-law non-linearity, a generalization of the standard Kerr law that applies to a wide range of real-world materials, including semiconductors and doped fibers. These findings have direct implications for modern optical communication technologies, including ultra-fast pulse transmission, all-optical switching, signal encoding, and the management of intensity dips and phase discontinuities in logic-based photonic circuits.

In summary, this research not only advances the analytical treatment of the CGLE under power-law nonlinearity but also opens new possibilities for engineering stable and controllable solitonic structures in nonlinear fiber optics. The interplay between nonlinearity, dispersion, and detuning, as captured by the model, reveals a rich landscape of wave behaviors that can be tuned through parameter engineering. Future studies may extend this framework to coupled systems, birefringent fibers, or fractional-order generalizations, further expanding its applicability in both theoretical and applied photonics.

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Research Article

Semi-discrete sampling operators acting on function spaces

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ABSTRACT. In this paper, we present an overview of recent advances in the study of the approximation properties of a family of semi-discrete sampling operators in several function spaces. We investigate their convergence properties in the space of continuous functions, providing an approximation result in the uniform norm and both quantitative and qualitative estimates. Here, we also establish a regularization result for functions in L^p-spaces. We then extend the study to the broader framework of Orlicz spaces, which allows the treatment of functions that are not necessarily continuous, as is often the case for real-world signals. In this setting, besides convergence, we also study the rate of approximation in terms of the φ-modulus of continuity defined by the modular functional. This unified approach yields approximation results in several particular cases, including Zygmund spaces, exponential-type spaces, and L^p-spaces. In the last setting, we are also able to achieve a sharper rate of convergence.

Keywords: Orlicz spaces, L^p-approximation, modular convergence, Durrmeyer sampling operators, Lipschitz classes, modulus of continuity.

2020 Mathematics Subject Classification: 41A25, 46E30, 47A58.

1. INTRODUCTION

We recall that Durrmeyer sampling operators, which are the focus of this overview, originate from the classical Bernstein polynomials, which were introduced in the context of polynomial approximation:

$$(\mathcal{B}_n f)(x) := \sum_{k=0}^{n} p_{n,k}(x)\, f\left(\frac{k}{n}\right), \quad p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}, \quad x \in [0,1].$$

These polynomials provided, in 1912, a constructive proof of the Weierstrass approximation theorem by algebraic polynomials in the space of continuous functions on $[0,1]$.

The Durrmeyer modification of Bernstein polynomials replaces the pointwise values $f(k/n)$ by an integral in which the same generating kernel of $\mathcal{B}_n$ appears:

$$(\mathcal{D}_n f)(x) := (n+1) \sum_{k=0}^{n} p_{n,k}(x) \int_0^1 p_{n,k}(u) f(u)\, du, \quad x \in [0,1],\ n \in \mathbb{N},$$

see, e.g., [33, 32, 20, 22, 21]. The literature on Durrmeyer-type operators is wide and includes several generalizations and variants, particularly concerning their approximation properties, convergence behavior, and asymptotic formulae [35, 36, 34, 38, 2, 1, 19].

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In this paper, we focus on the application of the Durrmeyer method to the generalized sampling series. As it is well-known, sampling-type operators were originally introduced to provide approximate versions of the celebrated Whittaker-Kotel'nikov-Shannon sampling theorem (see, e.g., [59, 43, 53]), which represents a rigorous mathematical model in signal processing. This theorem allows to reconstruct a continuous-time signal $f(t)$ at any instant t on the real line from a discrete set of sampled values $f(k/w)$, $k \in \mathbb{Z}$, $w > 0$, via an elegant interpolation formula (see, e.g., [16]).

Applying the Durrmeyer method to these sampling series leads to semi-discrete operators of the form

$$(1.1) \qquad (\mathcal{D}_w^{\varphi,\psi} f)(x) := \sum_{k\in\mathbb{Z}} \varphi(wx-k)\, w \int_{\mathbb{R}} \psi(wu-k) f(u)\, du, \quad x \in \mathbb{R},\ w > 0,$$

where φ and ψ are kernel functions satisfying standard moment conditions (see [11]). These operators are commonly known as Durrmeyer sampling operators based on φ and ψ. In this formulation, the pointwise evaluation of the function is replaced by a general convolution integral with the kernel ψ. Notice that ψ generates a Fejér-type approximate identity via $\psi_w(\cdot) := w\psi(w\cdot)$, $w > 0$. In particular, the operators in (1.1) can be seen as a result of a double convolution: A continuous convolution generated by ψ (called the continuous kernel) followed by a discrete one generated by φ (called the discrete kernel). By virtue of their semi-discrete nature, from (1.1) some important particular cases arise:

- If $\psi := \chi_{[0,1]}$, the characteristic function of $[0, 1]$, we obtain the Kantorovich sampling operators [28, 48, 6, 29, 30, 3, 31]:
$$(\mathcal{D}_w^{\varphi,\chi_{[0,1]}} f)(x) = \sum_{k\in\mathbb{Z}} \varphi(wx-k)\, w \int_{k/w}^{(k+1)/w} f(u)\, du =: (\mathcal{K}_w f)(x), \quad x \in \mathbb{R}.$$
- If $\psi := \delta$, the Dirac delta distribution, we recover the generalized sampling operators [14, 15, 57, 58, 42, 7]:
$$(\mathcal{D}_w^{\varphi,\delta} f)(x) = \sum_{k\in\mathbb{Z}} \varphi(wx-k) f\left(\frac{k}{w}\right) =: (\mathcal{G}_w f)(x), \quad x \in \mathbb{R}.$$

Operators in the form (1.1) were introduced by Bardaro and Mantellini to provide asymptotic expansions and Voronovskaja-type formulae for regular functions [11, 10, 12]. In particular, under suitable moment-type conditions, for $f \in C^r_{loc}(\mathbb{R})$, that is the space of r-times locally continuously differentiable functions on $\mathbb{R}$, the following local expansion holds:

$$(\mathcal{D}_w^{\varphi,\psi} f)(x) = \sum_{j=0}^{r} \frac{f^{(j)}(x)}{j!\, w^j} \sum_{\nu=0}^{j} \binom{j}{\nu} \widetilde{m}_{j-\nu}(\psi) m_\nu(\varphi) + o(w^{-r}), \quad \text{as } w \to +\infty,\ x \in \mathbb{R},$$

where $\widetilde{m}_\nu(\cdot)$ and $m_\nu(\cdot)$, $\nu \in \mathbb{N}$, denote the continuous and the discrete algebraic moments, respectively (see, e.g., Section 2). Some years later, Costarelli, Piconi and Vinti studied the convergence properties of (1.1) in more general function spaces, such as in $C(\mathbb{R})$ and in the Orlicz setting, also providing quantitative and qualitative results through suitable moduli of continuity [24, 25]; later, they also treated the multivariate case [23]. Several developments have followed, including the nonlinear version of operators in (1.1) [56], their variational properties [4], as well as extensions to weighted spaces [5] and to exponential variants of the operators [8, 18, 41]. In addition, Costarelli et al. investigated the regularization properties of (1.1) using a distributional approach, where they also obtained the distributional Fourier transform of the operators [26]. The same authors then studied higher-order approximation, proving direct estimates in terms of higher-order moduli of smoothness in L^p-spaces, along with inverse approximation results [27]. More recently, Sharma and Gupta investigated the convergence behavior of compositions of Durrmeyer sampling operators [54].

In this work, we provide an overview of some of the main results concerning the operators (1.1). We discuss convergence properties, rates of convergence, and regularization, ranging from continuous functions to the general Orlicz setting. In Section 2, we recall key tools such as the algebraic and absolute moments of the kernels and moduli of continuity in $C(\mathbb{R})$ and L^p settings. Section 3 presents the main results. In the space of continuous functions (Section 3.1), we show pointwise and uniform convergence, providing quantitative estimates in terms of the classical modulus of continuity, qualitative estimates for functions in suitable Lipschitz classes, and a regularization result showing how the operators can regularize a general function even when it is not necessarily continuous. Then, we consider the convergence properties of the operators in the more general context of Orlicz spaces (Section 3.2), which generalize L^p-spaces. Here, we provide a modular convergence theorem, from which convergence in several functional spaces, such as Zygmund or exponential-type spaces, can be deduced. Moreover, quantitative estimations are presented in terms of the modulus of continuity in Orlicz spaces, using an integral decay condition on the kernels. A special attention is devoted to the L^p-case, where sharper convergence rates can be achieved (Subsection 3.2.1).

The paper ends with final conclusions and some further developments.

2. Basic notions and preliminaries

We introduce the following notation.

For $1 \leq p \leq +\infty$, let $L^p(\mathbb{R})$ denote the usual Lebesgue space of measurable functions $f : \mathbb{R} \to \mathbb{R}$, equipped with the norm

$$\|f\|_p := \begin{cases} \left(\displaystyle\int_{\mathbb{R}} |f(x)|^p \, dx \right)^{1/p}, & 1 \leq p < +\infty, \\ \operatorname{ess\,sup}_{x \in \mathbb{R}} |f(x)|, & p = +\infty. \end{cases}$$

Moreover, we denote by $C(\mathbb{R})$ the subspace of $L^\infty(\mathbb{R})$ consisting of all bounded and uniformly continuous functions, where $\|\cdot\|_\infty$ coincides with the usual sup-norm.

Now, let us consider two functions $\varphi, \psi \in L^1(\mathbb{R})$, with φ being bounded in a neighborhood of the origin. We define the discrete and continuous algebraic moments of φ and ψ of order ν, respectively, as follows:

$$m_\nu(\varphi, u) := \sum_{k \in \mathbb{Z}} \varphi(u-k)(k-u)^\nu, \quad \widetilde{m}_\nu(\psi) := \int_{\mathbb{R}} u^\nu \psi(u) \, du, \quad u \in \mathbb{R}, \ \nu \in \mathbb{N}_0,$$

and the discrete and continuous absolute moments as

$$M_\nu(\varphi) := \sup_{u \in \mathbb{R}} \sum_{k \in \mathbb{Z}} \left|\varphi(u-k)\right| |u-k|^\nu, \quad \widetilde{M}_\nu(\psi) := \int_{\mathbb{R}} |u|^\nu \, |\psi(u)| \, du, \quad \nu > 0,$$

respectively. In particular, if the following basic condition holds

$$m_0(\varphi, u) = \widetilde{m}_0(\psi) = 1, \tag{2.2}$$

we call the functions φ and ψ as discrete and continuous kernel, respectively.

We observe that if $\mu, \nu > 0$ with $\mu \leq \nu$, then $M_\nu(\varphi) < +\infty$ implies $M_\mu(\varphi) < +\infty$, and similarly $\widetilde{M}_\nu(\psi) < +\infty$ implies $\widetilde{M}_\mu(\psi) < +\infty$. In particular, if φ has compact support, then $M_\nu(\varphi) < +\infty$ for every $\nu \geq 0$.

Moreover, throughout the present paper we always assume the natural moment-condition that

$$M_0(\varphi) < +\infty.$$

A first immediate consequence of this is that the operators $\mathcal{D}_w^{\varphi,\psi}$ are well-defined for every $f \in L^\infty(\mathbb{R})$, and satisfy

$$|(\mathcal{D}_w^{\varphi,\psi} f)(x)| \leq M_0(\varphi) \, \|\psi\|_1 \, \|f\|_\infty, \quad x \in \mathbb{R}. \tag{2.3}$$

Therefore, the sampling Durrmeyer operators are bounded linear operators mapping $L^\infty(\mathbb{R})$ into itself.

By a concise way, we denote by $X^p(\mathbb{R}) := L^p(\mathbb{R})$ if $1 \leq p < +\infty$ and by $X^\infty(\mathbb{R}) := C(\mathbb{R})$ if $p = +\infty$. We now recall the definition of the modulus of continuity in the space $X^p(\mathbb{R})$, $1 \leq p \leq +\infty$. (see, e.g., [17]).

Definition 2.1. *For $f \in X^p(\mathbb{R})$, the modulus of continuity is defined by*

$$\omega_p(f, \delta) := \sup_{|h| \leq \delta} \|f(\cdot + h) - f(\cdot)\|_p, \qquad \delta > 0.$$

It is well-known that for any $f \in X^p(\mathbb{R})$ there holds $\omega_p(f, \delta) \leq \omega_p(f, \delta')$ for every $0 < \delta \leq \delta'$ and the following useful inequality holds

$$\omega_p(f, \lambda\delta) \leq (1 + \lambda)\omega_p(f, \delta), \qquad \lambda, \delta > 0. \tag{2.4}$$

In this case, we can define the Lipschitz class of order $0 < \nu \leq 1$ in the space $X^p(\mathbb{R})$ for $1 \leq p \leq +\infty$ as

$$\mathrm{Lip}(\nu, p) := \Big\{ f \in X^p(\mathbb{R}) \ : \ \|f(\cdot + h) - f(\cdot)\|_p = O(h^\nu) \text{ as } h \to 0 \Big\}, \tag{2.5}$$

or, equivalently, the space of functions in $X^p(\mathbb{R})$ for which $\omega_p(f, \delta) = O(h^\nu)$ as $h \to 0$.

3. Main results

In the following, we present some recent advances in the study of approximation properties of Durrmeyer sampling operators, both in spaces of continuous functions and in the more general framework of Orlicz spaces. We show convergence results, quantitative and qualitative estimates for the rate of convergence, as well as regularization properties. A special focus is devoted to the case of L^p-spaces.

3.1. Approximation and regularization in $C(\mathbb{R})$.

A first objective in the space $C(\mathbb{R})$ is to study convergence in the uniform norm. Then, we establish the order of approximation through quantitative estimates based on the modulus of continuity introduced in Definition 2.1 for $p = +\infty$.

Lemma 3.1. *Let φ be a discrete kernel satisfying $M_\nu(\varphi) < +\infty$, for $\nu > 0$. Then for every $\gamma > 0$,*

$$\lim_{w \to +\infty} \sum_{|wx - k| > \gamma w} |\varphi(wx - k)| = 0,$$

uniformly with respect to $x \in \mathbb{R}$.

A proof of this lemma can be found, for instance, in [9]. Now, we are ready to claim the following pointwise and uniform convergence theorem in $C(\mathbb{R})$.

Theorem 3.1 ([24]). *Let $f \in L^\infty(\mathbb{R})$ and φ be a discrete kernel satisfying $M_\nu(\varphi) < +\infty$, for $\nu > 0$. Then, for every continuity point x of f, one has*

$$\lim_{w\to+\infty} (\mathcal{D}_w^{\varphi,\psi} f)(x) = f(x).$$

Moreover, if $f \in C(\mathbb{R})$, it follows that

$$\lim_{w\to+\infty} \|\mathcal{D}_w^{\varphi,\psi} f - f\|_\infty = 0.$$

Proof. We only prove the second statement, since the first one can be established by similar arguments. Fix $\varepsilon > 0$. By continuity of f, there exists $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ whenever $|x - y| < \delta$.

For a fixed $x \in \mathbb{R}$, using (2.2) we can write

$$\begin{aligned} |(\mathcal{D}_w^{\varphi,\psi} f)(x) - f(x)| &\le \sum_{k\in\mathbb{Z}} |\varphi(wx-k)|\, w \int_{\mathbb{R}} |\psi(wu-k)|\,|f(u)-f(x)|\,du \\ &= \left\{ \sum_{|wx-k|\le\frac{\delta}{2}w} + \sum_{|wx-k|>\frac{\delta}{2}w} \right\} |\varphi(wx-k)|\, w \int_{\mathbb{R}} |\psi(wu-k)|\,|f(u)-f(x)|\,du \\ &=: I_1 + I_2. \end{aligned}$$

For I_1, we note that if both $|wx - k| \le \frac{\delta}{2}w$ and $|wu - k| < \frac{\delta}{2}w$, then $|u - x| < \delta$, hence $|f(u) - f(x)| < \varepsilon$. Hence

$$I_{1,1} := \sum_{|wx-k|\le\frac{\delta}{2}w} |\varphi(wx-k)|\; w \int_{|wu-k|<\frac{\delta}{2}w} |\psi(wu-k)|\,|f(u)-f(x)|\,du < \varepsilon\, M_0(\varphi)\, \|\psi\|_1,$$

where the estimate follows from the change of variable $wu - k = y$ and the fact that $\psi \in L^1(\mathbb{R})$.

For the remaining part,

$$\begin{aligned} I_{1,2} :=& \sum_{|wx-k|\le\frac{\delta}{2}w} |\varphi(wx-k)|\; w \int_{|wu-k|>\frac{\delta}{2}w} |\psi(wu-k)|\,|f(u)-f(x)|\,du \\ \le& 2\|f\|_\infty M_0(\varphi) \int_{|y|\ge\frac{\delta}{2}w} |\psi(y)|\,dy, \end{aligned}$$

which tends to zero as $w \to \infty$, since $\psi \in L^1(\mathbb{R})$. Thus, there exists $w > 0$ sufficiently large such that $I_{1,2} \le 2\|f\|_\infty M_0(\varphi)\,\varepsilon$.

For I_2, a similar argument gives

$$I_2 \le 2\|f\|_\infty \|\psi\|_1 \sum_{|wx-k|>\frac{\delta}{2}w} |\varphi(wx-k)| < 2\|f\|_\infty \|\psi\|_1\, \varepsilon,$$

for $w > 0$ sufficiently large, as a consequence of Lemma 3.1.

Rearranging the estimates, we find

$$|(\mathcal{D}_w^{\varphi,\psi} f)(x) - f(x)| \lesssim \varepsilon,$$

for $w > 0$ sufficiently large. Since the bound is uniform in $x \in \mathbb{R}$, the claim is proved. □

After establishing convergence, we now turn to the study of the rate of convergence in the space $C(\mathbb{R})$, through a quantitative analysis using the classical modulus of continuity in this setting, and subsequently deducing the qualitative order in suitable Lipschitz classes. This analysis is summarized in the following result, whose proof is shown in [24].

Theorem 3.2 ([24])**.** *Assume that* φ *and* ψ *satisfy* $M_1(\varphi) + \widetilde{M}_1(\psi) < +\infty$, *and let* $f \in C(\mathbb{R})$. *Then,*

$$\|\mathcal{D}_w^{\varphi,\psi} f - f\|_\infty \leq C_{\varphi,\psi}\, \omega_\infty(f, 1/w)\,, \qquad w > 0,$$

where

$$C_{\varphi,\psi} := M_0(\varphi)\big(\widetilde{M}_0(\psi) + \widetilde{M}_1(\psi)\big) + M_1(\varphi)\widetilde{M}_0(\psi).$$

Moreover, if in addition $f \in \mathrm{Lip}(\nu, +\infty)$ *with* $0 < \nu \leq 1$, *then*

$$\|\mathcal{D}_w^{\varphi,\psi} f - f\|_\infty \leq C\, w^{-\nu}, \qquad w > 0,$$

where $C > 0$ *is a suitable absolute constant depending only on* φ, ψ *and* f.

As an example of discrete and continuous kernel satisfying the quantitative estimate, we consider the Bochner–Riesz kernel of order $\theta > 0$:

$$b_\theta(u) = \frac{2^\theta}{\sqrt{2\pi}}\, \frac{\Gamma(\theta+1)}{|u|^{\theta+1/2}}\, J_{\theta+1/2}(|u|), \qquad u \in \mathbb{R},$$

where J_λ denotes the Bessel function of order $\lambda > 0$ and Γ is the Euler gamma function (see Figure 1c). It holds that $b_\theta(u) = \mathcal{O}(|u|^{-\theta-1})$ as $|u| \to +\infty$, which implies $b_\theta \in L^1(\mathbb{R})$ and $M_\nu(b_\theta) < +\infty$ for every $0 \leq \nu < \theta$. Moreover, recalling that

$$\widehat{b_\theta}(v) = \begin{cases} (1-|v|^2)^\theta, & |v| \leq 1, \\ 0, & |v| > 1, \end{cases} \qquad v \in \mathbb{R},$$

we see that condition (2.2) is also fulfilled. This follows from the classical Poisson summation formula (see, e.g., [17]), since $\widehat{b_\theta}(k) = 1$ for all $k \in \mathbb{Z} \setminus \{0\}$ and $\widehat{b_\theta}(0) = 1$. Finally, we note that the finiteness of $M_1(b_\theta)$ holds assuming $\theta > 1$.

Corollary 3.1. *Let* $f \in C(\mathbb{R})$. *For any* $\theta > 0$, *there holds*

$$\lim_{w \to +\infty} \|\mathcal{D}_w^{b_\theta, b_\theta} f - f\|_\infty = 0.$$

In particular, if $\theta > 1$, *then*

$$\|\mathcal{D}_w^{b_\theta, b_\theta} f - f\|_\infty \leq C_{b_\theta}\, \omega_\infty(f, 1/w)\,, \qquad w > 0.$$

In this case, if in addition $f \in \mathrm{Lip}(\nu, +\infty)$ *with* $0 < \nu \leq 1$, *then*

$$\|\mathcal{D}_w^{b_\theta, b_\theta} f - f\|_\infty \leq C\, w^{-\nu}, \qquad w > 0,$$

where $C > 0$ *is a suitable absolute constant depending only on* b_θ *and* f.

Once convergence and the corresponding rate have been established, we now turn our attention to a regularization property. In fact, the following result shows that the smoothness of the operators $\mathcal{D}_w^{\varphi,\psi}$ is strongly influenced by the regularity of the discrete kernel φ. In particular, if φ is continuous and $f \in L^p(\mathbb{R})$ with $1 \leq p \leq +\infty$, then $\mathcal{D}_w^{\varphi,\psi} f$ is continuous for every $w > 0$. Hence, even when starting from a function f that is not continuous, the resulting Durrmeyer sampling operator yields a continuous function, meaning that these operators act as regularizers.

Theorem 3.3 ([26])**.** *Let* $f \in L^p(\mathbb{R})$, *with* $1 \leq p \leq +\infty$. *If* φ *is continuous (resp. uniformly continuous) and bounded, then* $\mathcal{D}_w^{\varphi,\psi} f$ *is continuous (resp. uniformly continuous) and bounded for every* $w > 0$.

Proof. Let $1 \le p < +\infty$ and fix $w > 0$. For $m \in \mathbb{N}^+$, define the truncated operators

$$d_w^m(x) := \sum_{|k| \le m} \left(w \int_{\mathbb{R}} \psi(wu-k) f(u)\, du \right) \varphi(wx-k), \quad x \in \mathbb{R}.$$

Then

$$\left| (\mathcal{D}_w^{\varphi,\psi} f)(x) - d_w^m(x) \right| \le \sum_{|k|>m} \left| w \int_{\mathbb{R}} \psi(wu-k) f(u)\, du \right| |\varphi(wx-k)|.$$

Fix $x \in \mathbb{R}$. By applying Jensen inequality, Fubini-Tonelli theorem, and using the boundedness of φ, one obtains

$$\left| (\mathcal{D}_w^{\varphi,\psi} f)(x) - d_w^m(x) \right| \le C\, w^{1/p} \|f\|_p \left(\sup_{u \in \mathbb{R}} \sum_{|k|>m} |\psi(wu-k)| \right)^{1/p},$$

for a suitable constant C depending only on φ and ψ. As $m \to +\infty$, the last term tends to zero, since $\sum_{k\in\mathbb{Z}} |\psi(wu-k)|$ converges uniformly in u by the finiteness of $M_0(\psi)$. Hence, d_w^m converges uniformly to $\mathcal{D}_w^{\varphi,\psi} f$ on $\mathbb{R}$ as $m \to +\infty$. Being the uniform limit of continuous functions by virtue of the continuity of φ, the operator $\mathcal{D}_w^{\varphi,\psi} f$ is itself continuous.

Moreover, by similar estimates one shows that there is a suitable constant $C' > 0$ depending only on φ and ψ such that

$$|(\mathcal{D}_w^{\varphi,\psi} f)(x)| \le C' w^{1/p} \|f\|_p, \quad x \in \mathbb{R},$$

so that $\mathcal{D}_w^{\varphi,\psi} f$ is also bounded. Therefore, $\mathcal{D}_w^{\varphi,\psi} f$ is continuous and bounded on $\mathbb{R}$ for $1 \le p < +\infty$.

For the case $p = +\infty$, one proceeds analogously by considering the symmetric kernel $\check{\psi}_w(\cdot) := w\psi(-w\cdot)$, leading to

$$\left| (\mathcal{D}_w^{\varphi,\psi} f)(x) - d_w^m(x) \right| \le \|\check{\psi}_w * f\|_\infty \sup_{x \in \mathbb{R}} \sum_{|k|>m} |\varphi(wx-k)|,$$

where the remainder again vanishes as $m \to +\infty$, since $M_0(\varphi) < +\infty$. Here, by $*$ we denote the usual convolution product. Moreover, by virtue of (2.3), we conclude that $\mathcal{D}_w^{\varphi,\psi} f$ is also continuous and bounded in this case.

To prove uniform continuity, observe that if $\varphi \in C(\mathbb{R})$, then each truncated operator d_w^m is uniformly continuous on $\mathbb{R}$, being a finite sum of uniformly continuous functions. Fix $\varepsilon > 0$. Since $d_w^m \to \mathcal{D}_w^{\varphi,\psi} f$ uniformly as $m \to +\infty$, there exists $m \in \mathbb{N}^+$ such that

$$\left| \mathcal{D}_w^{\varphi,\psi} f(x) - d_w^m(x) \right| < \tfrac{\varepsilon}{3}, \quad x \in \mathbb{R}.$$

Moreover, by uniform continuity of d_w^m, there exists $\delta > 0$ such that $|x-y| < \delta$ implies $|d_w^m(x) - d_w^m(y)| < \frac{\varepsilon}{3}$. Hence, for $|x-y| < \delta$,

$$\left| \mathcal{D}_w^{\varphi,\psi} f(x) - \mathcal{D}_w^{\varphi,\psi} f(y) \right| \le \left| \mathcal{D}_w^{\varphi,\psi} f(x) - d_w^m(x) \right| + \left| d_w^m(x) - d_w^m(y) \right| + \left| d_w^m(y) - \mathcal{D}_w^{\varphi,\psi} f(y) \right| < \varepsilon,$$

for m sufficiently large. Therefore, $\mathcal{D}_w^{\varphi,\psi} f$ is uniformly continuous. This completes the proof. □

For further examples and additional results on regularization in the case of continuous kernels, the reader may see [26].

3.2. **Approximation in $L^\eta(\mathbb{R})$.** Orlicz spaces originated in the 1930s as a natural extension of Lebesgue spaces [46, 47]. They are a remarkable example of modular spaces, introduced to generalize the notion of normed linear spaces. In what follows, we provide a brief overview of the fundamental notions related to Orlicz spaces [45, 44, 50, 51, 13, 37].

Let (Ω, Σ, μ) be a measure space with a σ-finite, complete measure μ. Let $\eta : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ be a φ-function, i.e., a non-decreasing continuous function with $\eta(0) = 0$, $\eta(u) > 0$ for $u > 0$ and $\lim_{u\to+\infty} \eta(u) = +\infty$. Denoting by $L^0(\Omega, \Sigma, \mu) = L^0(\Omega)$ the space of all real-valued, Σ-measurable functions on Ω, finite μ-almost everywhere, with equality μ-a.e., the functional $I^\eta : L^0(\Omega) \to \mathbb{R}_0^+$ defined as

$$I^\eta[f] := \int_\Omega \eta(|f(x)|)\, d\mu(x), \quad f \in L^0(\Omega),$$

is proved to be a modular (see, e.g., [13, 45]). Then, the Orlicz space generated by η is given by

$$L^\eta(\Omega) := \left\{ f \in L^0(\Omega) : I^\eta[\lambda f] < +\infty, \text{ for some } \lambda > 0 \right\}.$$

In cases where the modular I^η is convex (which holds when η itself is convex), we can define the so-called Luxemburg norm as follows

$$\|f\|_\eta := \inf \left\{ u > 0 : I^\eta \left[\frac{f}{u} \right] \le 1 \right\}.$$

In this case, the Orlicz space $L^\eta(\Omega)$ endowed with the norm $\|\cdot\|_\eta$ is a normed linear space (see Theorem 1.1 (b) of [13]). It is also a Banach space (see Theorem 3.3.7 (b) of [37]).

A φ-function η is said to satisfy the Δ_2-condition (in symbol, $\eta \in \Delta_2$) if there exists a constant $C_\eta > 0$ such that

$$\eta(u) \le C_\eta \eta(u) \quad \text{for all } u \ge 0.$$

Norm convergence can be characterized in terms of the modular, as the following lemma shows. This is often useful, since the exact value of the norm may be difficult to compute due to its definition as an infimum.

Lemma 3.2. *Let η be a convex φ-function. Then $\|f_k\|_\eta \to 0$ as $k \to +\infty$ if and only if*

$$\lim_{k\to+\infty} I^\eta[\lambda f_k] = 0 \quad \textit{for all } \lambda > 0.$$

A weaker and natural notion of convergence in $L^\varphi(\Omega)$ is the modular convergence. More precisely, a net of functions $(f_k)_{k>0} \subset L^\eta(\Omega)$ is said to converge modularly to a function $f \in L^\eta(\Omega)$, denoted by $f_k \xrightarrow{I^\eta} f$, if

$$\lim_{k\to+\infty} I^\eta[\lambda(f_k - f)] = 0, \quad \text{for some } \lambda > 0. \tag{3.6}$$

In some cases, modular convergence and norm convergence coincide.

Lemma 3.3. *Let η be a convex φ-function such that $\eta \in \Delta_2$. Then modular convergence and norm convergence are equivalent.*

It is also possible to define the notion of modular continuity in Orlicz spaces (see, e.g., [40]).

Definition 3.2. *Let η and ζ be a pair of φ-functions. A linear operator $T : L^\eta(\Omega) \to L^\zeta(\Omega)$ is said to be modularly continuous if $f_k \in L^\eta(\Omega)$, $f_k \xrightarrow{I^\eta} f$ as $k \to +\infty$, implies $T(f_k) \xrightarrow{I^\zeta} T(f)$ as $k \to +\infty$.*

Orlicz spaces include a wide range of functional spaces with several applications in various areas of pure and applied functional analysis, such as Fourier analysis, interpolation theory, partial differential equations, and distribution theory. A classical example of a φ-function satisfying the Δ_2-condition is $\varphi(u) = u^p$, with $u \geq 0$ and $1 \leq p < +\infty$, in which case $L^\varphi(\Omega) = L^p(\Omega)$, corresponding to the classical Lebesgue spaces. Here, $\| \cdot \|_\varphi = \| \cdot \|_p$.

Other examples of Orlicz spaces are, for instance, the interpolation spaces $L^\alpha \log^\beta L(\Omega)$ (also known as Zygmund spaces [60, 55]), which are useful from the applicative point of view. These spaces are generated by the φ-function

$$\eta_{\alpha,\beta}(u) := u^\alpha \log^\beta(u+e), \quad \alpha \geq 1,\ \beta > 0,\ u \geq 0.$$

The corresponding modular functional is defined as

$$I^{\eta_{\alpha,\beta}}[f] := \int_\Omega |f(u)|^\alpha \log^\beta(e + |f(u)|)\, du, \quad f \in M(\Omega).$$

It is well known that $\eta_{\alpha,\beta} \in \Delta_2$.

On the other hand, for the so-called exponential type spaces [52, 39], i.e., the Orlicz spaces generated by

$$\eta_\gamma(u) = e^{u^\gamma} - 1, \quad \gamma > 0,\ u \geq 0,$$

the Δ_2-condition is not satisfied, and consequently, modular and norm convergence are not equivalent. The related modular functional is given by

$$I^{\eta_\gamma}[f] := \int_\Omega \left(e^{|f(u)|^\gamma} - 1\right) du, \quad f \in M(\Omega).$$

We are now ready to present some approximation results for Durrmeyer sampling operators in the general framework of the Orlicz space $L^\eta(\Omega)$, with $\Omega = \mathbb{R}$, μ the Lebesgue measure, and Σ the σ-algebra of Lebesgue measurable subsets of $\mathbb{R}$. This provides a unified treatment, in this general context, for the study of the approximation properties of the class of operators considered here.

We begin with the following theorem, which shows that the Durrmeyer sampling operators are well defined on $L^\eta(\mathbb{R})$. Moreover, a modular convergence result holds in general; if the involved φ-function additionally satisfies the Δ_2-condition, a norm convergence result is also valid.

Theorem 3.4 ([24]). *Let η be a convex φ-function and $f \in L^\eta(\mathbb{R})$ be fixed. Moreover, let $M_0(\psi) < +\infty$. Then the following statements hold:*

(1) There exists $\lambda > 0$ such that

$$I^\eta[\lambda\, \mathcal{D}_w^{\varphi,\psi} f] \;\leq\; \frac{M_0(\psi)\|\varphi\|_1}{M_0(\varphi)\|\psi\|_1}\, I^\eta[\lambda\, M_0(\varphi)\, \|\psi\|_1\, f] < +\infty, \qquad w > 0.$$

In particular, the operators $\mathcal{D}_w^{\varphi,\psi}$ are well defined and belong to $L^\eta(\mathbb{R})$ for every $w > 0$.

(2) There exists $\lambda > 0$ such that

$$\lim_{w\to+\infty} I^\eta\big[\lambda\big(\mathcal{D}_w^{\varphi,\psi} f - f\big)\big] = 0.$$

(3) If $\eta \in \Delta_2$ then

$$\lim_{w\to+\infty} \|\mathcal{D}_w^{\varphi,\psi} f - f\|_\eta = 0.$$

For a proof, carried out through a unifying but technical direct modular estimate, the reader may refer to [24]. A key point in proving convergence is that the translated function $\Delta f_h = f(\cdot + h) - f(\cdot)$ converges modularly to 0, that is, $\Delta f_h \xrightarrow{I^\eta} 0$ as $h \to 0$.

As an immediate consequence of Theorem 3.4, the operators $\mathcal{D}_w^{\varphi,\psi}$ are also modularly continuous on $L^\eta(\mathbb{R})$ for every fixed $w > 0$, according to Definition 3.2. Indeed, since there exists $\lambda^* > 0$ with $I_\eta[\lambda^*(f - f_k)] \to 0$ as $k \to +\infty$, choosing $\lambda > 0$ so that $\lambda M_0(\varphi)\|\psi\|_1 \leq \lambda^*$, we have

$$I^\eta\big[\lambda(\mathcal{D}_w^{\varphi,\psi} f - \mathcal{D}_w^{\varphi,\psi} f_k)\big] \leq \frac{M_0(\psi)\|\varphi\|_1}{M_0(\varphi)\|\psi\|_1} I^\eta\big[\lambda^*(f - f_k)\big] \to 0, \quad k \to +\infty.$$

As example of convergence of the operators in the Orlicz setting, we consider the Fejér kernel, defined as

$$\mathrm{F}(u) = \frac{1}{2}\mathrm{sinc}^2\left(\frac{u}{2}\right), \quad u \in \mathbb{R},$$

where $\mathrm{sinc}(u) = \sin(\pi u)/\pi u$ if $u \in \mathbb{R} \setminus \{0\}$ and 1 if $u = 0$ (see Figure 1a). The kernel is bounded, non-negative on $\mathbb{R}$, and belongs to $L^1(\mathbb{R})$. Moreover, condition (2.2) is satisfied, being $\widehat{\mathrm{F}}(2k\pi) = 0$ for $k \in \mathbb{Z} \setminus \{0\}$ and $\widehat{\mathrm{F}}(0) = 1$, where

$$\widehat{\mathrm{F}}(v) = \begin{cases} 1 - |\frac{v}{\pi}|, & |v| \leq \pi, \\ 0, & |v| > \pi, \end{cases} \quad v \in \mathbb{R}.$$

In particular, $\|\mathrm{F}\|_1 = M_0(\mathrm{F}) = 1$. In this case, we obtain the following.

Corollary 3.2. *Let η be a convex φ-function and $f \in L^\eta(\mathbb{R})$ be fixed. Thus, there exists $\lambda > 0$ such that*

$$I^\eta\big[\lambda\, \mathcal{D}_w^{\mathrm{F},\mathrm{F}} f\big] \;\leq\; I^\eta\big[\lambda\, f\big], \qquad w > 0.$$

In particular, there exists $\lambda > 0$ such that

$$\lim_{w \to +\infty} I^\eta\big[\lambda\big(\mathcal{D}_w^{\mathrm{F},\mathrm{F}} f - f\big)\big] = 0.$$

Now, in order to study the rate of convergence, we need the following tool.

Definition 3.3. *Let η be a φ-function. The map $\omega_\eta : L^0(\mathbb{R}) \times \mathbb{R}^+ \to [0, +\infty]$ defined by*

$$\omega_\eta(f, \delta) := \sup_{|h| \leq \delta} I^\eta[f(h + \cdot) - f(\cdot)],$$

for $f \in L^0(\mathbb{R})$, is called the η-modulus of continuity.

Lemma 3.4. *Let η be a φ-function. Then for every function $f \in L^0(\mathbb{R})$ there exists $\lambda > 0$ such that*

$$\omega_\eta(\lambda f, \delta) \to 0 \quad \text{as } \delta \to 0^+.$$

For a proof, see Theorem 2.4 of [13].

As in the classical case, we now introduce the notion of Orlicz-Lipschitz classes defined in terms of the modular functional I^η. Let $\mathcal{T}$ be the class of measurable functions $\tau : \mathbb{R} \to [0, +\infty]$ such that $\tau(t) > 0$ for all $t \neq 0$.

Definition 3.4. *For a given $\tau \in \mathcal{T}$, we define the Orlicz-Lipschitz class as*

$$\mathrm{Lip}(\tau, \eta) := \left\{ f \in L^\eta(\mathbb{R}) : \exists \lambda > 0 \text{ with } I^\eta\big[\lambda\big(f(\cdot + h) - f(\cdot)\big)\big] = \mathcal{O}(\tau(h)),\ h \to 0 \right\},$$

where, for any two functions $f, g \in L^\eta(\mathbb{R})$,

$$f(t) = \mathcal{O}(g(t)), \quad t \to 0$$

means that there exist a constant $C > 0$ and some $\delta > 0$ such that $|f(t)| \leq C|g(t)|$ for $|t| \leq \delta$.

To obtain quantitative estimates in terms of the η-modulus of continuity, we require the following further condition on the kernels.

For any $0<\alpha<1$, we say that a function $\xi:\mathbb{R}\to\mathbb{R}$ satisfies the integral decay condition (α) if

$$(\alpha)\qquad w\int_{|u|>1/w^{\alpha}}|\xi(wu)|\,du\le Kw^{-\mu},\quad \text{as } w\to+\infty,$$

for suitable constants $K,\mu>0$ depending on α and ξ.

We are now ready to state the main results about the rate of convergence of Durrmeyer sampling operators in Orlicz spaces. From the quantitative estimate below, we directly deduce the qualitative order of approximation, assuming f belongs to a suitable Orlicz-Lipschitz class. In particular, we use the Lipschitz classes from Definition 3.4, with $\tau(h)=h^{\nu}$, $0<\nu\le 1$. In this case, we denote the resulting Lipschitz class simply by $\mathrm{Lip}(\nu,\eta)$.

Theorem 3.5 ([25]). *Let η be a convex φ-function and let $f\in L^{\eta}(\mathbb{R})$. Moreover, let $M_0(\psi)<+\infty$. Suppose $0<\alpha<1$ and that both kernels φ and ψ satisfy condition (α).*

Then there exist constants $K>0$ and $\mu>0$, depending on α,φ,ψ, such that

$$\begin{aligned}I^{\eta}\big[\lambda\big(\mathcal{D}_w^{\varphi,\psi}f-f\big)\big]\le&\left(\frac{M_0(\psi)\|\varphi\|_1+M_0(\varphi)\|\psi\|_1}{2M_0(\varphi)\|\psi\|_1}\right)\omega_{\eta}\big(2\lambda M_0(\varphi)\|\psi\|_1 f,w^{-\alpha}\big)\\&+K\left(\frac{M_0(\varphi)+M_0(\psi)}{2M_0(\varphi)\|\psi\|_1}\right)I^{\eta}[4\lambda M_0(\varphi)\|\psi\|_1 f]\,w^{-\mu},\end{aligned}$$

for $\lambda>0$ and every sufficiently large $w>0$. In particular, if $\lambda>0$ is small enough, the inequality implies the modular convergence of the Durrmeyer sampling operators $\mathcal{D}_w^{\varphi,\psi}f$ to f.

Moreover, if $f\in\mathrm{Lip}(\nu,\eta)$ with $0<\nu\le 1$, then there exist constants $C,\lambda>0$ such that

$$I^{\eta}\big[\lambda\big(\mathcal{D}_w^{\varphi,\psi}f-f\big)\big]\le Cw^{-\rho},$$

for all sufficiently large $w>0$, where $\rho:=\min\{\alpha\nu,\mu\}$.

Proof. Fix $\lambda>0$. By convexity of η, we can write

$$I^{\eta}\big[\lambda\big(D_w^{\varphi,\psi}f-f\big)\big]\le\tfrac{1}{2}(I_1+I_2),$$

where

$$I_1:=\int_{\mathbb{R}}\eta\left(2\lambda\left|(D_w^{\varphi,\psi}f)(x)-\sum_{k\in\mathbb{Z}}\varphi(wx-k)w\int_{\mathbb{R}}\psi(wu-k)f\big(u+x-\tfrac{k}{w}\big)\,du\right|\right)dx,$$

and

$$I_2:=\int_{\mathbb{R}}\eta\left(2\lambda\left|\sum_{k\in\mathbb{Z}}\varphi(wx-k)w\int_{\mathbb{R}}\psi(wu-k)f\big(u+x-\tfrac{k}{w}\big)\,du-f(x)\right|\right)dx.$$

For I_1, using Jensen inequality twice, the change of variable $y=x-k/w$ and Fubini-Tonelli theorem, we obtain

$$I_1\le\frac{M_0(\psi)}{M_0(\varphi)\|\psi\|_1}\int_{\mathbb{R}}w|\varphi(wy)|\left(\int_{\mathbb{R}}\eta(2\lambda M_0(\varphi)\|\psi\|_1\,|f(u+y)-f(u)|)\,du\right)dy.$$

Let $0<\alpha<1$ of condition (α) be fixed. Splitting the integral into the intervals $|y|\le 1/w^{\alpha}$ and $|y|>1/w^{\alpha}$, we obtain

$$I_1\le\frac{M_0(\psi)\|\varphi\|_1}{M_0(\varphi)\|\psi\|_1}\omega_{\eta}\big(2\lambda M_0(\varphi)\|\psi\|_1 f,w^{-\alpha}\big)+\frac{M_0(\psi)}{M_0(\varphi)\|\psi\|_1}I^{\eta}[4\lambda M_0(\varphi)\|\psi\|_1 f]\,K_{\varphi}w^{-\mu_{\varphi}},$$

thanks to the convexity of η, the translation invariance of

$$I^\eta[4\lambda M_0(\varphi)\|\psi\|_1 f(\cdot)] = I^\eta[4\lambda M_0(\varphi)\|\psi\|_1 f(\cdot + y)]$$

for every $y \in \mathbb{R}$ and condition (α) for φ.

Let us now estimate I_2. After the change of variable $t = u - \frac{k}{w}$, we obtain

$$I_2 = \int_{\mathbb{R}} \eta\left(2\lambda\left|\sum_{k\in\mathbb{Z}} \varphi(wx-k)\right|\left|\int_{\mathbb{R}} \psi(wt)\left[f(t+x) - f(x)\right]dt\right|\right) dx.$$

Applying Jensen inequality and using (2.2), this gives

$$I_2 \leq \frac{w}{\|\psi\|_1} \int_{\mathbb{R}} |\psi(wt)|\, I^\eta[2\lambda M_0(\varphi)\|\psi\|_1\, |f(t+\cdot) - f(\cdot)|]\, dt.$$

Splitting the integral into $|t| \leq 1/w^\alpha$ and $|t| > 1/w^\alpha$, we get

$$I_2 \leq \omega_\eta\big(2\lambda M_0(\varphi)\|\psi\|_1 f, w^{-\alpha}\big) + \frac{1}{\|\psi\|_1} I^\eta[4\lambda M_0(\varphi)\|\psi\|_1 f]\, K_\psi w^{-\mu_\psi},$$

since ψ also satisfies (α).

Combining the above two estimates, and setting

$$K := \max\{K_\varphi, K_\psi\}, \qquad \mu := \min\{\mu_\varphi, \mu_\psi\},$$

we obtain the desired inequality.

The second part of the statement, namely the qualitative estimate, directly follows from the above quantitative bound together with Definition 3.4. This concludes the proof. □

We highlight that condition (α) is satisfied by several examples of kernels, not necessarily only those with compact support, for which it becomes trivially verified. For instance, in the case of the Bochner Riesz kernel b_θ, for any fixed $0 < \alpha < 1$, one can take $K = \widetilde{M}_\nu(b_\theta)$ and $\mu = \nu(1-\alpha)$ for every $0 \leq \nu < \theta$. This follows from the fact that $b_\theta(u) = O(|u|^{-\theta-1})$ as $|u| \to +\infty$. Similarly, for the Fejér kernel F, one can show that $K = \widetilde{M}_\nu(\mathrm{F})$ and $\mu = \nu(1-\alpha)$ for every $0 \leq \nu < 1$.

3.2.1. *Sharp analysis in L^p-spaces.* Herein, we focus on the particular case of L^p-spaces, where we are able to get a sharper order of approximation with respect to the one obtained in the Orlicz case.

As a consequence of Theorem 3.4, we obtain the following result regarding the well-definedness and convergence properties of the Durrmeyer sampling operators in L^p-spaces.

Corollary 3.3 ([24]). *Let $M_0(\varphi) < +\infty$. The following assertions hold:*

(1) For every $f \in L^p(\mathbb{R})$, with $1 \leq p < +\infty$, we have

$$\|\mathcal{D}_w^{\varphi,\psi} f\|_p \leq M_0(\psi)^{\frac{1}{p}} M_0(\varphi)^{\frac{p-1}{p}} \|\varphi\|_1^{\frac{1}{p}} \|\psi\|_1^{\frac{p-1}{p}} \|f\|_p, \qquad w > 0.$$

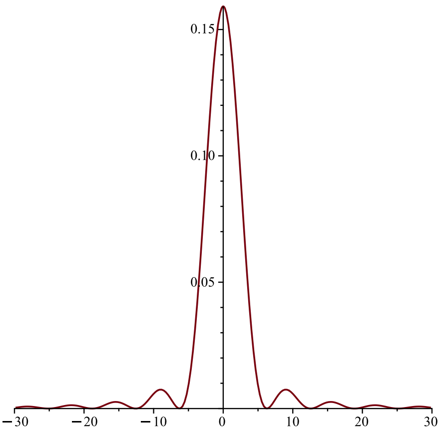
In particular, $\mathcal{D}_w^{\varphi,\psi} f$ is well defined in $L^p(\mathbb{R})$ and belongs to $L^p(\mathbb{R})$ whenever $f \in L^p(\mathbb{R})$.

(2) For every $f \in L^p(\mathbb{R})$, with $1 \leq p < +\infty$, we have

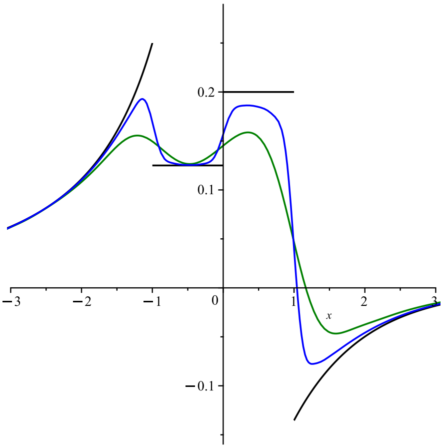
$$\lim_{w\to+\infty} \|\mathcal{D}_w^{\varphi,\psi} f - f\|_p = 0.$$

For a graphical representation, the reader can see Figures 1b and 1d.

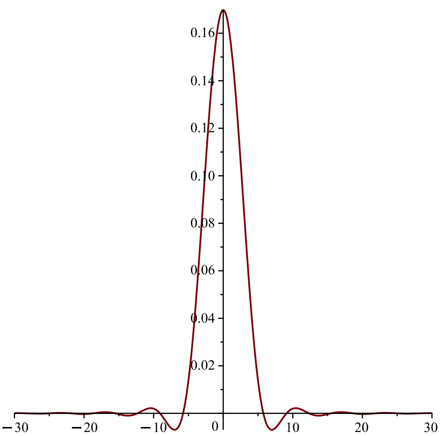
Regarding the rate of convergence, in the case of L^p-spaces we are able to achieve a sharper order compared with the one stated in the general Orlicz-space setting (Theorem 3.5).



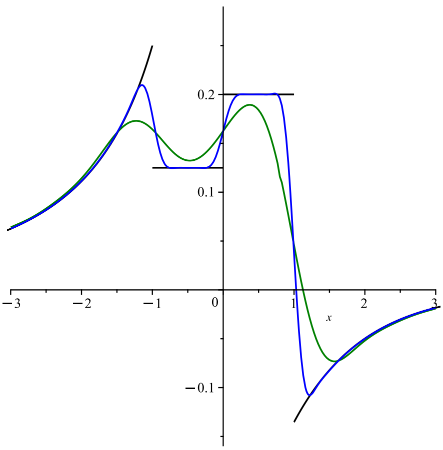
(A) Graph of the Fejér kernel.



(B) Approximation of a discontinuous function (black line) using Durrmeyer sampling operators generated by the Fejér kernel $\mathcal{D}_w^{\mathrm{F},\mathrm{F}}$ for $w = 10$ (green line) and $w = 30$ (blue line).



(C) Graph of the Bochner-Riesz kernel of order $\theta = 2$.



(D) Approximation of the same discontinuous function (black line) using Durrmeyer sampling operators generated by the Bochner-Riesz kernel (of order $\theta = 2$) $\mathcal{D}_w^{b_2,b_2}$ for $w = 10$ (green line) and $w = 30$ (blue line).

FIGURE 1. Comparison between Fejér and Bochner-Riesz kernels and their corresponding Durrmeyer approximations of a discontinuous function in $L^1(\mathbb{R})$.

Corollary 3.4 ([25]). *Let $1 \leq p < +\infty$ and let φ, ψ be such that*

$$\widetilde{M}_p(\varphi) + \widetilde{M}_p(\psi) < +\infty.$$

Then, for every $f \in L^p(\mathbb{R})$, we have the quantitative estimate

$$\|\mathcal{D}_w^{\varphi,\psi} f - f\|_p \leq M_0(\varphi)\,(2\|\psi\|_1)^{\frac{p-1}{p}} \times \left[\left(M_0(\psi)\|\varphi\|_1 + \frac{M_p(\varphi)}{M_0(\varphi)} \right)^{\frac{1}{p}} + (\|\psi\|_1 + M_p(\psi))^{\frac{1}{p}} \right] \omega_p\big(f, w^{-1}\big),$$

for all sufficiently large $w > 0$.

Moreover, if $f \in \mathrm{Lip}(\nu, p)$ with $0 < \nu \leq 1$, then there exists a constant $C > 0$ such that

$$\|\mathcal{D}_w^{\varphi,\psi} f - f\|_p \leq M_0(\varphi)\,(2\|\psi\|_1)^{\frac{p-1}{p}} \left[\left(M_0(\psi)\|\varphi\|_1 + \frac{M_p(\varphi)}{M_0(\varphi)} \right)^{\frac{1}{p}} + (\|\psi\|_1 + M_p(\psi))^{\frac{1}{p}} \right] C w^{-\nu},$$

for every sufficiently large $w > 0$.

For a detailed proof, see [25]. The sharper quantitative estimates in Corollary 3.4 are obtained using a more direct approach than in the general setting of Theorem 3.5. This is possible thanks to the specific inequality, recalled in (2.4), satisfied by the L^p-modulus of smoothness ω_p, which does not hold in general in Orlicz spaces. As a result, the convergence rates in the Lipschitz classes $\mathrm{Lip}(\nu, p)$, introduced in (2.5) for $1 \leq p < +\infty$, are improved.

Conclusions and Future Developments

In this work, we have provided a general overview of some of the main approximation results for semi-discrete Durrmeyer-type operators, considering the problem in different functional settings. In the space of continuous functions, we established a theorem of pointwise and uniform convergence, supported by both quantitative and qualitative estimates, together with a regularization result in which the discrete kernel plays a crucial role. We then moved to the framework of Orlicz spaces, where we proved convergence results and obtained quantitative estimates through suitable moduli of continuity defined in terms of the modular. A particular focus has been devoted to the case of L^p-spaces, where a sharper order of approximation can be achieved.

The study of convergence, approximation order, and regularization by a distributional approach has opened new directions of research. In particular, we investigated quantitative estimates based on higher-order moduli of smoothness, which allow us to establish conditions ensuring higher-order convergence in L^p-spaces, of order $r > 1$ for some integer r. We also considered the delicate problem of inverse approximation, that is, deducing regularity properties of a function from the knowledge of the convergence rate of Durrmeyer operators in L^p-spaces. By combining direct and inverse results, we obtained a full characterization of the well-known generalized Lipschitz classes in the L^p-setting [27].

Finally, regarding the Orlicz framework, our research is moving towards the study of convergence of operators in Sobolev-Orlicz spaces, which extend Sobolev spaces in the same way that Orlicz spaces extend L^p-spaces [49]. In this context, we are investigating simultaneous approximation results for the derivatives of the operators to the derivatives of the functions, which represent a natural line of further developments.

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